TRANSMUTATION AND BOSONISATION OF QUASI-HOPF ALGEBRAS

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ABSTRACT. Let H be a quasitriangular quasi-Hopf algebra, we construct a braided group $\underline{H}$ in the quasiassociative category of left H-modules. Conversely, given any braided group B in this category, we construct a quasi-Hopf algebra $B \rtimes H$ in the category of vector spaces. We generalise the transmutation and bosonisation theory of [10] to the quasi case. As examples, we bosonise the octonion algebra to an asooiciative one, obtain the twisted quantum double $D^{\phi}(G)$ of a finite group as a bosonisation, and obtain its transmutation. Finally, we show that $\underline{H} \rtimes H \cong H_{\mathcal{R}} \bowtie H$ as quasi-Hopf algebras.

1. INTRODUCTION

If H is a quasitriangular Hopf algebra, it is known that there exists a Hopf algebra $\underline{H}$ in the category ${}_H\mathcal{M}$ of left H-modules, a construction known as 'transmutation' [6]. Following Majid, we refer to Hopf algebras in braided categories as 'braided groups'. Conversely, given a braided group B in the category ${}_H\mathcal{M}$, there exists a Hopf algebra $B \rtimes H$ in the category of vector spaces [10], a construction known as 'bosonisation'. We recall the required theory in section 2.

In sections 3 and 4 we generalise these results to H a quasitriangular quasi-Hopf algebra [4]. In this case the associativity constraint in the category ${}_H\mathcal{M}$ is no longer trivial, it now depends on the associator of the quasi-Hopf algebra. Nevertheless, we show that the theory of [10] follows through. One has a transmutation $\underline{H}$ as a braided group in ${}_H\mathcal{M}$. In [1] it was shown that for any algebra B in ${}_H\mathcal{M}$ there is an associative algebra $B \rtimes H$. We extend this to B a braided group in ${}_H\mathcal{M}$ and obtain a quasi-Hopf algebra $B \rtimes H$. One also has, for example a one to one correspondence between braided B-modules in ${}_H\mathcal{M}$ and left $B \rtimes H$-modules in the category of vector spaces. We consider the examples of the twisted quantum double $D^{\phi}(G)$ introduced in [3], and the octonions in the form [8].

It is known for quasitriangular Hopf algebras that there exists an isomorphism between $\underline{H} \rtimes H$ and the twisted tensor product $H_{\mathcal{R}} \bowtie H$; in section 5 we prove that when H is a quasitriangular quasi-Hopf algebra, there is a quasi-Hopf algebra isomorphism $\chi : \underline{H} \rtimes H \to H_{\mathcal{R}} \bowtie H$.

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2. Preliminaries

2.1. **Quasi-Hopf Algebras.** Let k be a commutative field. A *quasi-bialgebra*, [4], over k is $(H, \Delta, \varepsilon, \phi)$ where H is a unital associative algebra over k, $\Delta : H \to H \otimes H$ is an algebra homomorphism such that

$$(\mathrm{id} \otimes \Delta)\Delta = \phi(\Delta \otimes \mathrm{id})\Delta\phi^{-1},$$

and the axiom for the counit ε, an algebra homomorphism, are as usual. The element $\phi \in H \otimes H \otimes H$, called the *Drinfeld associator*, or *associator*, that controls the noncoassociativity is invertible, and is required to be a counital 3-cocyle, i.e.

$$(1 \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})\phi(\phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\phi(\Delta \otimes \mathrm{id} \otimes \mathrm{id})\phi,$$

and $(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})(\phi) = 1\otimes1\otimes1$. H is a *quasi-Hopf algebra* if there exists a convolution invertible algebra anti-homomorphism $S : H \to H$, called the antipode, together with elements $\alpha, \beta \in H$ such that,

$$\begin{gathered}
S(h_{(1)})\alpha h_{(2)} = \varepsilon(h)\alpha, \\
h_{(1)}\beta S(h_{(2)}) = \varepsilon(h)\beta, \\
X^1\beta S(X^2)\alpha X^3 = 1, \\
S(x^1)\alpha x^2\beta S(x^3) = 1,
\end{gathered}$$

for all $h \in H$, where $\phi = X^1 \otimes X^2 \otimes X^3$ is written in capital letters, and $\phi^{-1} = x^1 \otimes x^2 \otimes x^3$ is written in lower case letters. For brevity, the sum notation for the coproduct and the Drinfeld associator has been suppressed. The antipode is uniquely determined up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, for any invertible $U \in H$. Following from this, we can, without loss of generality, assume $\varepsilon(\alpha) = \varepsilon(\beta) = 1$.

For Hopf algebras it is known that the antipode is a coalgebra anti-homomorphism; in the case of quasi-Hopf algebras this is true only up to a twist, i.e. there exists $f \in H \otimes H$ such that

$$f\Delta(S(h))f^{-1} = S(\Delta^{op}(h))$$

for all $h \in H$.

Following [4], define $\gamma, \delta \in H \otimes H$ by

$$\begin{aligned}
\gamma &= S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \\
\delta &= B^1\beta S(B^4) \otimes B^2\beta S(B^3)
\end{aligned}$$

where

$$\begin{aligned}
A^1 \otimes A^2 \otimes A^3 \otimes A^4 &= (\phi \otimes 1)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi^{-1}) \\
B^1 \otimes B^2 \otimes B^3 \otimes B^4 &= (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi)(\phi^{-1} \otimes 1)
\end{aligned}$$

Denote f^{-1} by g, then f, g are given by the formulae

$$f = (S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2\beta S(x^3))$$

$$g = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{op}(x^3))$$

Further, f satisfies $f\Delta(\alpha) = \gamma$, $\Delta(\beta)g = \delta$, and we note

$$\Delta(X^1)\delta(S \otimes S)(\Delta^{op}(X^2))\gamma\Delta(X^3) = 1$$

$$(S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2)\delta(S \otimes S)(\Delta^{op}(x^3)) = 1$$

It is useful to define elements $q = q^1 \otimes q^2 = \sum X^1 \otimes S^{-1}(\alpha X^3)X^2$ and $p = p^1 \otimes p^2 = \sum x^1 \otimes x^2\beta S(x^3)$ in $H \otimes H$. Then, for all $h \in H$,

$$\Delta(h_{(1)})p(1 \otimes S(h_{(2)})) = p(h \otimes 1),$$

$$(1 \otimes S^{-1}(h_{(2)}))q\Delta(h_{(1)}) = (h \otimes 1)q,$$

$$\Delta(q^1)p(1 \otimes S(q^2)) = 1 \otimes 1,$$

$$(1 \otimes S^{-1}(p^2))q\Delta(p^1) = 1 \otimes 1.$$

The quasi-Hopf algebra $(H, \Delta, \varepsilon, S, \alpha, \beta, \phi)$ is *quasitriangular* [4] if there is an invertible element $R \in H \otimes H$ such that,

$$(\Delta \otimes \mathrm{id})(R) = \phi_{312}R_{13}\phi_{132}^{-1}R_{23}\phi,$$

$$(\mathrm{id} \otimes \Delta)(R) = \phi_{231}^{-1}R_{13}\phi_{213}R_{12}\phi^{-1},$$

$$\Delta^{op}(h) = R\Delta(h)R^{-1},$$

for all $h \in H$. Writing $\phi = \sum X^1 \otimes X^2 \otimes X^3$, then $\phi_{ijk} \in H \otimes H \otimes H$ has X^1 in the i-th position, X^2 in the j-th position and X^3 in the k-th position, for example, $\phi_{312} = X^2 \otimes X^3 \otimes X^1$. Similarly for $\phi = \sum x^1 \otimes x^2 \otimes x^3$. The inverse [2] is given by

$$R^{-1} = X^1\beta S(Y^2R^{(1)}x^1X^2)\alpha Y^3x^3{X^3}_{(2)} \otimes Y^1R^{(2)}x^2{X^3}_{(1)}$$

As for quasitriangular Hopf algebras, $(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1 \otimes 1$. Further, the above relations imply the quasi-Yang-Baxter equation

$$R_{12}\phi_{312}R_{13}\phi_{132}^{-1}R_{23}\phi = \phi_{321}R_{23}\phi_{231}^{-1}R_{13}\phi_{213}R_{12}.$$

2.2. **Monoidal Categories.** A *monoidal category* is $(\mathcal{C}, \otimes, I, \Phi, l, r)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called the tensor product, and I is a fixed unit object. Further, $\mathcal{C}$ is equipped with a natural transformation, called the *associativity constraint*, consisting of functorial isomorphisms $\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for all $U, V, W \in \mathcal{C}$, obeying the well-known pentagon condition. Finally, $\mathcal{C}$ is equipped with natural transformations l, r, consisting of functorial isomorphisms $l_V : V \to I \otimes V$, $r_V : V \to V \otimes I$ for each $V \in \mathcal{C}$, obeying the triangle condition. Here l and r are called the *left* and *right unit constraints* respectively.

An object V in a monoidal category $\mathcal{C}$ is *rigid* if there exists an object V^* and morphisms $ev_V : V^* \otimes V \to I$ and $coev_V : I \to V \otimes V^*$ in $\mathcal{C}$, such that

$$r_V^{-1}(\mathrm{id}_V \otimes ev_V)\Phi_{V,V^*,V}(coev_V \otimes \mathrm{id}_V)l_V = \mathrm{id}_V,$$

$$l_{V^*}^{-1}(ev_V \otimes \mathrm{id}_{V^*})\Phi^{-1}_{V^*,V,V^*}(\mathrm{id}_{V^*} \otimes coev_V)r_{V^*} = \mathrm{id}_{V^*}.$$

The monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ is rigid. A *braided* category [11] is a monoidal category $(\mathcal{C}, \otimes)$ equipped with a natural transformation consisting of functorial isomorphisms $\Psi_{V,W} : V \otimes W \to W \otimes V$ for all $V, W \in \mathcal{C}$, called a *braiding*, obeying the well-known hexagon conditions. We will use the notation of [9].

Example 2.1. [4] Let H be a unital algebra, then the category ${}_H\mathcal{M}$ of left H-modules consists of objects, the vector spaces V on which H acts, and morphisms, the linear maps f which commute with the action of H, i.e. $f(h \triangleright v) = h \triangleright f(v)$ for all $v \in V$ and $h \in H$. If H is a quasi-bialgebra, then $\otimes$, defined by $h \triangleright (v \otimes w) = h_{(1)} \triangleright v \otimes h_{(2)} \triangleright w$, and

$$\Phi_{U,V,W}((u \otimes v) \otimes w) = X^1 \triangleright u \otimes (X^2 \triangleright v \otimes X^3 \triangleright w)$$

for all $u \in U, v \in V, w \in W$ where $U, V, W \in {}_H\mathcal{M}$, makes ${}_H\mathcal{M}$ into a monoidal category. If H is a quasi-triangular quasi-Hopf algebra, then ${}_H\mathcal{M}$ is a braided monoidal category with the braiding defined by

$$\Psi_{U,V}(u \otimes v) = R^{(2)} \triangleright v \otimes R^{(1)} \triangleright u$$

for all $u \in U, v \in V$. Finally, this category is rigid with $(h \triangleright f)(v) = f(S(h) \triangleright v)$ for all $v \in V$, $f \in V^*$ and $h \in H$, and

$$ev(f \otimes v) = f(\alpha \triangleright v)$$

$$coev = \sum_a \beta \triangleright e_a \otimes f^a$$

where $\{e_a\}$ is a basis for V and $\{f^a\}$ a dual basis. We refer to [9] for details.

2.3. **Hopf Algebras in Braided Categories.** An algebra in a monoidal category $\mathcal{C}$ is an object B of $\mathcal{C}$ equipped with a multiplication morphism $B \otimes B \to B$ and a unit morphism $\underline{1} \to B$, obeying the usual associativity and unit axioms, but now as morphisms in $\mathcal{C}$, and where $B \otimes B$ is the tensor product in the catgeory. A bialgebra in a braided category is an algebra B in the category equipped with algebra morphisms $\underline{\Delta} : B \to B \otimes B$ and $\underline{\varepsilon} : B \to \underline{1}$ in $\mathcal{C}$ which form a coalgebra in the category. Further, if there is a morphism $\underline{S} : B \to B$ in $\mathcal{C}$ obeying the usual antipode axioms, then B is a Hopf algebra in the braided category $\mathcal{C}$. The Hopf algebra B is called a *braided Hopf algebra* or *braided group*, [12].

Following [9], we consider monoidal categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{D}$ braided, and functors $F, V \otimes F : \mathcal{C} \to \mathcal{D}$. Suppose there is an object $B \in \mathcal{D}$ such that for all $V \in \mathcal{D}$, $Mor(V, B) \cong Nat(V \otimes F, F)$ by functorial isomorphisms θ_V. Let

$$\alpha = \{\alpha_M : B \otimes F(M) \to F(M) | M \in \mathcal{C}\}$$

be the natural transformation corresponding to the identity morphism id_B in $Mor(V, B)$. Then, using α, and the braiding we get induced maps

$$\theta_V^n : Mor(V, B^{\otimes n}) \to Nat(V \otimes F^n, F^n)$$

and we assume these are bijections. This is called the *representability assumption for modules*. Then, using these bijections, we can define a multiplication, a unit, a coproduct, a counit and an antipode for B.

For example, note that $\alpha_M(\mathrm{id} \otimes \alpha_M)\Phi_{B,B,F(M)} : (B \otimes B) \otimes F(M) \to F(M)$ is a natural transformation in $Nat(B \otimes B \otimes F, F)$, and hence corresponds to a unique map $B \otimes B \to B$ under $\theta^{-1}_{B \otimes B}$, which must be the multiplication on B. We will require the following theorem,

Theorem 2.2. [9] *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with $\mathcal{D}$ braided, and $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor satisfying the representability assumption for modules. Then B, as above, is a bialgebra in $\mathcal{D}$. If $\mathcal{D}$ is rigid, then B is a Hopf-algebra in $\mathcal{D}$.*

This theorem with $\mathcal{C} =_H \mathcal{M}$, for H a quasitriangular Hopf algebra, and $F = \mathrm{id}$ is used to reconstruct a braided group, $\underline{H}$ [10]. Taking the monoidal category of B-modules in the braided category of H-modules and the forgetful functor to Vec, and reconstructing, we obtain an ordinary Hopf algebra, which is the categorical theory of bosonisation. We now do the same when H is a quasitriangular quasi-Hopf algebra.

3. Transmutation of Quasi-Hopf Algebras

Let H be a quasitriangular quasi-Hopf algebra, B_L be the vector space H with the left regular action, and let B be the vector space H viewed as an object of ${}_H\mathcal{M}$ via the left adjoint action $h \triangleright g = h_{(1)} g S(h_{(2)})$ for all $h, g \in H$. In the notation of the above theorem, we consider the case when $\mathcal{C} = \mathcal{D} = {}_H\mathcal{M}$, and $F = \mathrm{id}$.

First we define $\theta_V : Mor(V,B) \to Nat(V\otimes \mathrm{id}, \mathrm{id})$ for $V \in {}_H\mathcal{M}$ as follows. Given $\psi \in Mor(V,B)$, we define $\xi \in Nat(V\otimes \mathrm{id}, \mathrm{id})$ by

$$\xi_M(v\otimes m) = \theta_V(\psi)_M(v\otimes m) = q^1\psi(v)S(q^2)\triangleright m,$$

where $\triangleright$ is the action of H on M as an object in the category ${}_H\mathcal{M}$. We have to check that each $\xi_M : V\otimes M \to M$ is a morphism in the category if ψ is.

$$\begin{aligned}
\xi_M(h\triangleright(v\otimes m)) &= \xi_M(h_{(1)}\triangleright v\otimes h_{(2)}\triangleright m)\\
&= q^1\psi(h_{(1)}\triangleright v)S(q^2)h_{(2)}\triangleright m\\
&= q^1(h_{(1)}\triangleright\psi(v))S(q^2)h_{(2)}\triangleright m\\
&= q^1h_{(1,1)}\psi(v)S(S^{-1}(h_{(2)})q^2h_{(1,2)})\triangleright m\\
&= hq^1\psi(v)S(q^2)\triangleright m\\
&= h\triangleright(q^1\psi(v)S(q^2)\triangleright m)\\
&= h\triangleright(\xi_M(v\otimes m))
\end{aligned}$$

Conversely, we define $\theta_V^{-1} : Nat(V\otimes \mathrm{id}, \mathrm{id}) \to Mor(V,B)$ for $V \in {}_H\mathcal{M}$ as follows. Given $\xi \in Nat(V\otimes \mathrm{id}, \mathrm{id})$, we define $\psi \in Mor(V,B)$ by

$$\psi(v) = \theta_V^{-1}(\xi)(v) = \xi_{B_L}(p^1\triangleright v\otimes p^2),$$

for all $v \in V$. Now,

$$\begin{aligned}
h\triangleright\psi(v) &= h\triangleright\xi_{B_L}(p^1\triangleright v\otimes p^2)\\
&= h_{(1)}\xi_{B_L}(p^1\triangleright v\otimes p^2)S(h_{(2)})\\
&= \xi_{B_L}(h_{(1)}\triangleright(p^1\triangleright v\otimes p^2))S(h_{(2)})\\
&= \xi_{B_L}(h_{(1,1)}p^1\triangleright v\otimes h_{(1,2)}\triangleright p^2)S(h_{(2)})\\
&= \xi_{B_L}(h_{(1,1)}p^1\triangleright v\otimes h_{(1,2)}p^2)S(h_{(2)})\\
&= \xi_{B_L}(h_{(1,1)}p^1\triangleright v\otimes h_{(1,2)}p^2S(h_{(2)}))\\
&= \xi_{B_L}(p^1h\triangleright v\otimes p^2)\\
&= \xi_{B_L}(p^1\triangleright(h\triangleright v)\otimes p^2)\\
&= \psi(h\triangleright v).
\end{aligned}$$

It is straightfoward to check these two processes are mutually inverse. The natural transformation corresponding to the identity morphism on B is $\alpha = \{\alpha_M | M \in {}_H\mathcal{M}\}$, where each $\alpha_M : B\otimes M \to M$ is given by

$$\alpha_M(b\otimes m) = \theta_B(\mathrm{id}_B)_M(b\otimes n) = q^1bS(q^2)\triangleright m$$

Theorem 3.1. *Every quasitriangular quasi-Hopf algebra H has a braided group analogue $\underline{H}$ in ${}_H\mathcal{M}$ and is given by*

$$\underline{m}(b\otimes b') = q^1(x^1\triangleright b)S(q^2)x^2b'S(x^3)$$
$$\underline{\eta}(1) = \beta$$
$$\underline{\Delta}(b) = x^1X^1b_{(1)}g^1S(x^2R^{(2)}y^3X^3{}_{(2)})\otimes x^3R^{(1)}\triangleright y^1X^2b_{(2)}g^2S(y^2X^3{}_{(1)})$$
$$\underline{\varepsilon}(b) = \varepsilon(b)$$
$$\underline{S}(b) = X^1R^{(2)}x^2\beta S(q^1(X^2R^{(1)}x^1\triangleright b)S(q^2)X^3x^3)$$

Proof. Let $M\in {}_H\mathcal{M}$. We have that $\alpha_M(\mathrm{id}\otimes\alpha_M)\Phi_{B,B,M}:(B\otimes B)\otimes M\to M$ is a natural transformation in $Nat(B\otimes B\otimes \mathrm{id},\mathrm{id})$, and hence corresponds to a unique map $\underline{m}:B\otimes B\to B$ under $\theta^{-1}_{B\otimes B}$. Let $\xi_M=\alpha_M(\mathrm{id}\otimes\alpha_M)\Phi_{B,B,M}$, then for all $b,b'\in B$ and $m\in M$,

$$\begin{aligned}\xi_M((b\otimes b')\otimes m) &= \alpha_M(\mathrm{id}\otimes\alpha_M)\Phi_{B,B,M}((b\otimes b')\otimes m)\\ &= \alpha_M(\mathrm{id}\otimes\alpha_M)(X^1\triangleright b\otimes(X^2\triangleright b'\otimes X^3\triangleright m))\\ &= \alpha_M(X^1\triangleright b\otimes q^1(X^2\triangleright b')S(q^2)X^3\triangleright m)\\ &= Q^1(X^1\triangleright b)S(Q^2)q^1(X^2\triangleright b')S(q^2)X^3\triangleright m\end{aligned}$$

where $Q^1\otimes Q^2$ is another copy of $q=q^1\otimes q^2$. Then,

$$\begin{aligned}\underline{m}(b\otimes b') &= \theta^{-1}_{B\otimes B}(\xi)(b\otimes b')\\ &= \xi_{B_L}(p^1\triangleright(b\otimes b')\otimes p^2)\\ &= \xi_{B_L}((p^1{}_{(1)}\triangleright b\otimes p^1{}_{(2)}\triangleright b')\otimes p^2)\\ &= Q^1(X^1p^1{}_{(1)}\triangleright b)S(Q^2)q^1(X^2p^1{}_{(2)}\triangleright b')S(q^2)X^3\triangleright p^2\\ &= Q^1(X^1p^1{}_{(1)}\triangleright b)S(Q^2)q^1(X^2p^1{}_{(2)}\triangleright b')S(q^2)X^3p^2\\ &= Q^1(X^1x^1{}_{(1)}\triangleright b)S(Q^2)q^1(X^2x^1{}_{(2)}\triangleright b')S(q^2)X^3x^2\beta S(x^3)\\ &= Q^1(x^1X^1\triangleright b)S(Q^2)q^1(x^2{}_{(1)}y^1X^2\triangleright b')S(q^2)x^2{}_{(2)}y^2X^3{}_{(1)}\\ &\quad \beta S(X^3{}_{(2)})S(X^3y^3)\\ &= Q^1(x^1X^1\triangleright b)S(Q^2)q^1(x^2{}_{(1)}y^1X^2\triangleright b')S(q^2)x^2{}_{(2)}y^2\varepsilon(X^3)\\ &\quad \beta S(x^3y^3)\\ &= Q^1(x^1\triangleright b)S(Q^2)q^1(x^2{}_{(1)}y^1\triangleright b')S(q^2)x^2{}_{(2)}y^2\beta S(x^3y^3)\\ &= Q^1(x^1\triangleright b)S(Q^2)q^1x^2_{(1,1)}(y^1\triangleright b')S(S^{-1}(x^2{}_{(2)}q^2x^2_{(1,2)})y^2\\ &\quad \beta S(x^3y^3)\\ &= Q^1(x^1\triangleright b)S(Q^2)x^2q^1(y^1\triangleright b')S(q^2)y^2\beta S(x^3y^3)\\ &= X^1(x^1\triangleright b)S(S^{-1}(\alpha X^3)X^2)x^2q^1(y^1\triangleright b')S(q^2)y^2\\ &\quad \beta S(x^3y^3)\\ &= X^1(x^1\triangleright b)S(X^2)\alpha X^3x^2q^1(y^1\triangleright b')S(q^2)y^2\\ &\quad \beta S(x^3y^3)\\ &= X^1x^1{}_{(1)}bS(X^2x^1{}_{(2)})\alpha X^3x^2q^1(y^1\triangleright b')S(q^2)y^2\\ &\quad \beta S(x^3y^3)\end{aligned}$$

$$\begin{aligned}
&= X^1 x^1{}_{(1)} b S(X^2 x^1{}_{(2)}) \alpha X^3 x^2 q^1 (p^1 \triangleright b') S(q^2) p^2 S(x^3)\\
&= y^1 X^1 b S(y^2{}_{(1)} x^1 X^2) \alpha y^2{}_{(2)} x^2 X^3{}_{(1)} q^1 (p^1 \triangleright b') S(q^2)\\
&\quad p^2 S(y^3 x^3 X^3{}_{(2)})\\
&= y^1 X^1 b S(x^1 X^2) S(y^2{}_{(1)}) \alpha y^2{}_{(2)} x^2 X^3{}_{(1)} q^1 (p^1 \triangleright b') S(q^2)\\
&\quad p^2 S(y^3 x^3 X^3{}_{(2)})\\
&= y^1 X^1 b S(x^1 X^2) \varepsilon(y^2) \alpha x^2 X^3{}_{(1)} q^1 (p^1 \triangleright b') S(q^2)\\
&\quad p^2 S(y^3 x^3 X^3{}_{(2)})\\
&= X^1 b S(x^1 X^2) \alpha x^2 X^3{}_{(1)} q^1 (p^1 \triangleright b') S(q^2) p^2 S(x^3 X^3{}_{(2)})\\
&= X^1 b S(x^1 X^2) \alpha x^2 X^3{}_{(1)} q^1 p^1{}_{(1)} b' S(S^{-1}(p^2) q^2 p^1{}_{(2)}) S(x^3 X^3{}_{(2)})\\
&= X^1 b S(x^1 X^2) \alpha x^2 X^3{}_{(1)} b' S(x^3 X^3{}_{(2)})\\
&= X^1 x^1{}_{(1)} b S(X^2 x^1{}_{(2)}) \alpha X^3 x^2 b' S(x^3)\\
&= q^1 (x^1 \triangleright b) S(q^2) x^2 b' S(x^3)
\end{aligned}$$

So, for all $b, b' \in B$, the multiplication is defined by

$$\underline{m}(b \otimes b') = q^1 (x^1 \triangleright b) S(q^2) x^2 b' S(x^3).$$

The antipode is determined by $\underline{S}(b) = \theta_B^{-1}(\xi)(b)$, where

$\xi_M = r_M^{-1}(M \otimes ev_M)\Phi_{M,M^*,M}((M \otimes \alpha_{M^*}) \otimes M)(\Phi_{M,B,M^*} \otimes M)((\Psi_{B,M} \otimes M^*) \otimes M)(\Phi^{-1}_{B,M,M^*} \otimes M)((B \otimes coev_M) \otimes M)(r_B \otimes M)$.

So,

$$\xi_M(b \otimes m) = Q^1 X^1 R^{(2)} x^2 \beta S(q^1 (X^2 R^{(1)} x^1 \triangleright b) S(q^2) X^3 x^3) S(Q^2) \triangleright m$$

hence,

$$\begin{aligned}
\underline{S}(b) &= \theta_B^{-1}(\xi)(b)\\
&= \xi_{B_L}(p^1 \triangleright b \otimes p^2)\\
&= Q^1 X^1 R^{(2)} x^2 \beta S(q^1 (X^2 R^{(1)} x^1 p^1 \triangleright b) S(q^2) X^3 x^3) S(Q^2) \triangleright p^2\\
&= Q^1 X^1 R^{(2)} P^2 S(q^1 (X^2 R^{(1)} P^1 p^1 \triangleright b) S(q^2) X^3 x^3) S(Q^2)\\
&= Q^1 X^1 R^{(2)} p^1{}_{(1)(2)} P^2 S(p^1{}_{(2)}) S(q^1 (X^2 R_{(1)} p^1{}_{(1)(1)} P^1 \triangleright b) S(q^2) X^3) S(Q^2) p^2\\
&= Q^1 X^1 p^1{}_{(1)(1)} R^{(2)} P^2 S(q^1 (X^2 p^1{}_{(1)(2)} R^{(1)} P^1 \triangleright b) S(q^2) X^3 p^1{}_{(2)}) S(Q^2) p^2\\
&= Q^1 p^1{}_{(1)} X^1 R^{(2)} P^2 S(q^1 (p^1{}_{(2)(1)} X^2 R^{(1)} P^1 \triangleright b) S(q^2) p^1{}_{(2)(2)} X^3) S(Q^2) p^2\\
&= Q^1 p^1{}_{(1)} X^1 R^{(2)} P^2 S(p^1{}_{(2)} q^1 (X^2 R^{(1)} P^1 \triangleright b) S(q^2) X^3) S(Q^2) p^2\\
&= Q^1 p^1{}_{(1)} X^1 R^{(2)} P^2 S(q^1 (X^2 R^{(1)} P^1 \triangleright b) S(q^2) X^3) S(S^{-1}(p^2) Q^2 p^1{}_{(1)})\\
&= X^1 R^{(2)} p^2 S(q^1 (X^2 R^{(1)} p^1 \triangleright b) S(q^2) X^3)
\end{aligned}$$

So the antipode is defined as

$$\underline{S}(b) = X^1 R^{(2)} p^2 S(q^1 (X^2 R^{(1)} p^1 \triangleright b) S(q^2) X^3)$$

The reconstructed $\underline{\Delta}$ is characterised by

$$\begin{aligned}\alpha_{M\otimes N}\Phi_{B,M,N} =& (\alpha_M \otimes \alpha_N)\Phi^{-1}_{B,M,B\otimes N}(B\otimes \Phi_{M,B,N})(B\otimes(\Psi_{B,M}\otimes N))\\ &(B\otimes \Phi^{-1}_{B,M,N})\Phi_{B,B,M\otimes N}(\underline{\Delta}\otimes(M\otimes N))\Phi_{B,M,N}.\end{aligned}$$

Let $b\in B, m\in M, n\in N$, then

$$\begin{aligned}\alpha_{M\otimes N}\Phi_{B,M,N}((b\otimes m)\otimes n) &= \alpha_{M\otimes N}(X^1\triangleright b\otimes(X^2\triangleright m\otimes X^3\triangleright n))\\ &= q^1(X^1\triangleright b)S(q^2)\triangleright(X^2\triangleright m\otimes X^3\triangleright n)\\ &= (q^1(X^1\triangleright b)S(q^2))_{(1)}X^2\triangleright m\otimes(q^1(X^1\triangleright b)S(q^2))_{(2)}X^3\triangleright n\end{aligned}$$

and,

$$\begin{aligned}&(\alpha_M\otimes\alpha_N)\dots\Phi_{B,M,N}((b\otimes m)\otimes n)\\ &= q^1(y^1Y^1\triangleright(X^1\triangleright b)_{\underline{(1)}})S(q^2)y^2Z^1R^{(2)}x^2Y^3{}_{(1)}X^2\triangleright m\otimes\\ &\quad Q^1(y^3{}_{(1)}Z^2R^{(1)}x^1Y^2\triangleright(X^1\triangleright b)_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Z^3x^3Y^3{}_{(2)}X^3\triangleright n\end{aligned}$$

Since these are equal for all $b\in B, m\in M, n\in N$, we have

$$\begin{aligned}&(q^1(X^1\triangleright b)S(q^2))_{(1)}X^2\otimes(q^1(X^1\triangleright b)S(q^2))_{(2)}X^3\\ &= q^1(y^1Y^1\triangleright(X^1\triangleright b)_{\underline{(1)}})S(q^2)y^2Z^1R^{(2)}x^2Y^3{}_{(1)}X^2\otimes\\ &\quad Q^1(y^3{}_{(1)}Z^2R^{(1)}x^1Y^2\triangleright(X^1\triangleright b)_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Z^3x^3Y^3{}_{(2)}X^3\end{aligned}$$

Which can be further simplified to

$$\begin{aligned}\Delta(q^1bS(q^2)) = q^1(y^1X^1\triangleright b_{\underline{(1)}})S(q^2)y^2Y^1R^{(2)}x^2X^3{}_{(1)}\\ \otimes Q^1(y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright b_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\end{aligned}\tag{3.1}$$

We can check that $\underline{\Delta}(b) = x^1X^1b_{(1)}g^1S(x^2R^{(2)}y^3X^3{}_{(2)})\otimes x^3R^{(1)}\triangleright y^1X^2b_{(2)}g^2S(y^2X^3{}_{(1)})$ satisfies this identity as follows.

$$\begin{aligned}&q^1(y^1X^1\triangleright b_{\underline{(1)}})S(q^2)y^2Y^1R^{(2)}x^2X^3{}_{(1)}\otimes Q^1(y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright b_{\underline{(2)}})S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\\ &= q^1(y^1X^1\triangleright w^1A^1b_{(1)}g^1S(w^2R'^{(2)}z^3A^3{}_{(2)}))S(q^2)y^2Y^1R^{(2)}x^2X^3{}_{(1)}\\ &\quad\otimes Q^1(y^3{}_{(1)}Y^2R^{(1)}x^1X^2w^3R'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\\ &= W^1y^1{}_{(1)}\underline{X^1{}_{(1)}w^1}A^1b_{(1)}g^1S(W^2y^1{}_{(2)}\underline{X^1{}_{(2)}w^2}R'^{(2)}z^3A^3{}_{(2)})\alpha W^3y^2R^{(2)}x^2\underline{X^3{}_{(1)}}\\ &\quad\otimes Q^1(y^3{}_{(1)}Y^2R^{(1)}x^1\underline{X^2w^3}R'^{(1)}\triangleright z^1A^2b_{(2)}g^2S(z^2A^3{}_{(1)}))S(Q^2)y^3{}_{(2)}Y^3x^3\underline{X^3{}_{(2)}}\\ &= W^1y^1{}_{(1)}w^1X^1A^1b_{(1)}g^1S(W^2y^1{}_{(2)}w^2T^1X^2{}_{(1)}R'^{(2)}z^3A^3{}_{(2)})\alpha\end{aligned}$$

$$
\begin{aligned}
&W^3 y^2 \underline{Y^1 R^{(2)} x^2 w^3{}_{(2)(1)}} T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(y^3{}_{(1)} \underline{Y^2 R^{(1)} x^1 w^3{}_{(1)}} T^2 X^2{}_{(2)} R'^{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2)\\
&y^3{}_{(2)} \underline{Y^3 x^3 w^3{}_{(2)(2)}} T^3{}_{(2)} X^3{}_{(2)}\\
=\;&\underline{W^1 y^1{}_{(1)} w^1} X^1 A^1 b_{(1)} g^1 S(\underline{W^2 y^1{}_{(2)} w^2} T^1 X^2{}_{(1)} R'^{(2)} z^3 A^3{}_{(2)})\alpha\\
&\underline{W^3 y^2 w^3{}_{(1)}} Y^1 R^{(2)} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(\underline{y^3{}_{(1)} w^3{}_{(2)(1)}} Y^2 R^{(1)} x^1 T^2 X^2{}_{(2)} R'^{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2)\\
&\underline{y^3{}_{(2)} w^3{}_{(2)(2)}} Y^3 x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&y^1 X^1 A^1 b_{(1)} g^1 S(y^2{}_{(1)} w^1 T^1 X^2{}_{(1)} R'^{(2)} z^3 A^3{}_{(2)})\alpha y^2{}_{(2)} w^2 Y^1 R^{(2)} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(y^3{}_{(1)} w^3{}_{(1)} Y^2 R^{(1)} x^1 T^2 X^2{}_{(2)} R'^{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2)\\
&y^3{}_{(2)} w^3{}_{(2)} Y^3 x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(w^1 T^1 X^2{}_{(1)} R'^{(2)} z^3 A^3{}_{(2)})\alpha w^2 Y^1 R^{(2)} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(w^3{}_{(1)} Y^2 R^{(1)} x^1 T^2 X^2{}_{(2)} R'^{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2)\\
&w^3{}_{(2)} Y^3 x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(w^1 T^1 \underline{X^2{}_{(1)} R'^{(2)}} z^3 A^3{}_{(2)})\alpha w^2 Y^1 R^{(2)} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(w^3{}_{(1)} Y^2 R^{(1)} x^1 T^2 \underline{X^2{}_{(2)} R'^{(1)}} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) w^3{}_{(2)} Y^3 x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(\underline{w^1} T^1 R'^{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha \underline{w^2 Y^1} R^{(2)} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(\underline{w^3{}_{(1)} Y^2} R^{(1)} x^1 T^2 R'^{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) \underline{w^3{}_{(2)} Y^3} x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(w^1 t^1 T^1 R'^{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha w^2 \underline{t^2{}_{(1)} R^{(2)}} x^2 T^3{}_{(1)} X^3{}_{(1)}\\
&\otimes Q^1(w^3 \underline{t^2{}_{(2)} R^{(1)}} x^1 T^2 R'^{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) t^3 x^3 T^3{}_{(2)} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(w^1 \underline{t^1 T^1} R'^{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha w^2 R^{(2)} \underline{t^2{}_{(2)} x^2 T^3{}_{(1)}} X^3{}_{(1)}\\
&\otimes Q^1(w^3 R^{(1)} \underline{t^2{}_{(1)} x^1 T^2} R'^{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) \underline{t^3 x^3 T^3{}_{(2)}} X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(w^1 T^1 \underline{x^1{}_{(1)} R'^{(2)}} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha w^2 R^{(2)} T^3 x^2 X^3{}_{(1)}\\
&\otimes Q^1(w^3 R^{(1)} T^2 \underline{x^1{}_{(2)} R'^{(1)}} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) x^3 X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(\underline{w^1 T^1 R'^{(2)}} x^1{}_{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha \underline{w^2 R^{(2)} T^3} x^2 X^3{}_{(1)}\\
&\otimes Q^1(\underline{w^3 R^{(1)} T^2 R'^{(1)}} x^1{}_{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) x^3 X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(R^{(2)}{}_{(1)} Y^2 x^1{}_{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha R^{(2)}{}_{(2)} Y^3 x^2 X^3{}_{(1)}\\
&\otimes Q^1(R^{(1)} Y^1 x^1{}_{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) x^3 X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(Y^2 x^1{}_{(2)} X^2{}_{(2)} z^3 A^3{}_{(2)})\alpha Y^3 x^2 X^3{}_{(1)}\\
&\otimes Q^1(Y^1 x^1{}_{(1)} X^2{}_{(1)} \triangleright z^1 A^2 b_{(2)} g^2 S(z^2 A^3{}_{(1)}))S(Q^2) x^3 X^3{}_{(2)}\\
=\;&X^1 A^1 b_{(1)} g^1 S(Y^2 \underline{x^1{}_{(2)} X^2{}_{(2)} z^3} A^3{}_{(2)})\alpha Y^3 x^2 X^3{}_{(1)}\\
&\otimes W^1 Y^1{}_{(1)} \underline{x^1{}_{(1)(1)} X^2{}_{(1)(1)} z^1} A^2 b_{(2)} g^2 S(W^2 Y^1{}_{(2)} \underline{x^1{}_{(1)(2)} X^2{}_{(1)(2)} z^2} A^3{}_{(1)})\alpha W^3 x^3 X^3{}_{(2)}\\
=\;&\underline{X^1 A^1} b_{(1)} g^1 S(Y^2 z^3 x^1{}_{(2)(2)} \underline{X^2{}_{(2)(2)} A^3{}_{(2)}})\alpha Y^3 x^2 \underline{X^3{}_{(1)}}
\end{aligned}
$$

$$
\begin{aligned}
&\otimes W^1{Y^1}_{(1)}z^1{x^1}_{(1)}\underline{{X^2}_{(1)}A^2}b_{(2)}g^2S(W^2{Y^1}_{(2)}z^2{x^1}_{(2)(1)}\underline{{X^2}_{(2)(1)}{A^3}_{(1)}})\alpha \\
&\quad W^3x^3\underline{{X^3}_{(2)}} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(\underline{Y^2z^3}{x^1}_{(2)(2)}{y^2}_{(2)}{X^3}_{(1)(2)}{A^2}_{(2)})\alpha\underline{Y^3}x^2{y^3}_{(1)}{X^3}_{(2)(1)}{A^3}_{(1)} \\
&\quad \otimes W^1\underline{{Y^1}_{(1)}z^1}{x^1}_{(1)}y^1X^2{A^1}_{(2)}b_{(2)}g^2S(W^2\underline{{Y^1}_{(2)}z^2}{x^1}_{(2)(1)}{y^2}_{(1)}{X^3}_{(1)(1)}{A^2}_{(1)})\alpha \\
&\quad W^3x^3{y^3}_{(2)}{X^3}_{(2)(2)}{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(T^2{Y^2}_{(2)}{x^1}_{(2)(2)}{y^2}_{(2)}{X^3}_{(1)(2)}{A^2}_{(2)})\alpha T^3Y^3x^2{y^3}_{(1)}{X^3}_{(2)(1)}{A^3}_{(1)} \\
&\quad \otimes W^1Y^1{x^1}_{(1)}y^1X^2{A^1}_{(2)}b_{(2)}g^2S(W^2T^1{Y^2}_{(1)}{x^1}_{(2)(1)}{y^2}_{(1)}{X^3}_{(1)(1)}{A^2}_{(1)})\alpha \\
&\quad W^3x^3{y^3}_{(2)}{X^3}_{(2)(2)}{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(T^2\underline{{Y^2}_{(2)}{x^1}_{(2)(2)}{y^2}_{(2)}}{X^3}_{(1)(2)}{A^2}_{(2)})\alpha T^3\underline{Y^3x^2{y^3}_{(1)}}{X^3}_{(2)(1)}{A^3}_{(1)} \\
&\quad \otimes W^1\underline{Y^1{x^1}_{(1)}y^1}X^2{A^1}_{(2)}b_{(2)}g^2S(W^2T^1\underline{{Y^2}_{(1)}{x^1}_{(2)(1)}{y^2}_{(1)}}{X^3}_{(1)(1)}{A^2}_{(1)})\alpha \\
&\quad W^3\underline{x^3{y^3}_{(2)}}{X^3}_{(2)(2)}{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(T^2{y^2}_{(1)(2)}\underline{{x^1}_{(2)}{X^3}_{(1)(2)}}{A^2}_{(2)})\alpha T^3{y^2}_{(2)}\underline{x^2{X^3}_{(2)(1)}}{A^3}_{(1)} \\
&\quad \otimes W^1y^1X^2{A^1}_{(2)}b_{(2)}g^2S(W^2T^1{y^2}_{(1)(1)}\underline{{x^1}_{(1)}{X^3}_{(1)(1)}}{A^2}_{(1)})\alpha W^3y^3\underline{x^3{X^3}_{(2)(2)}}{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(\underline{T^2{y^2}_{(1)(2)}{X^3}_{(1)(1)(2)}}{x^1}_{(2)}{A^2}_{(2)})\alpha\underline{T^3{y^2}_{(2)}{X^3}_{(1)(2)}}x^2{A^3}_{(1)} \\
&\quad \otimes W^1y^1X^2{A^1}_{(2)}b_{(2)}g^2S(W^2\underline{T^1{y^2}_{(1)(1)}{X^3}_{(1)(1)(1)}}{x^1}_{(1)}{A^2}_{(1)})\alpha W^3y^3{X^3}_{(2)}x^3{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(T^2{x^1}_{(2)}{A^2}_{(2)})\alpha T^3x^2{A^3}_{(1)} \\
&\quad \otimes W^1y^1X^2{A^1}_{(2)}b_{(2)}g^2S(W^2y^2{X^3}_{(1)}T^1{x^1}_{(1)}{A^2}_{(1)})\alpha W^3y^3{X^3}_{(2)}x^3{A^3}_{(2)} \\
&= X^1{A^1}_{(1)}b_{(1)}g^1S(T^2{x^1}_{(2)}{A^2}_{(2)})\alpha T^3x^2{A^3}_{(1)} \\
&\quad \otimes X^2{A^1}_{(2)}b_{(2)}g^2S({X^3}_{(1)}T^1{x^1}_{(1)}{A^2}_{(1)})\alpha{X^3}_{(2)}x^3{A^3}_{(2)} \\
&= {A^1}_{(1)}b_{(1)}g^1S(T^2{x^1}_{(2)}{A^2}_{(2)})\alpha T^3x^2{A^3}_{(1)}\otimes{A^1}_{(2)}b_{(2)}g^2S(T^1{x^1}_{(1)}{A^2}_{(1)})\alpha x^3{A^3}_{(2)} \\
&= {X^1}_{(1)}b_{(1)}g^1S({X^2}_{(1)})S(Y^2{x^1}_{(2)})\alpha Y^3x^2{X^3}_{(1)} \\
&\quad \otimes{X^1}_{(2)}b_{(2)}g^2S({X^2}_{(1)})S(Y^1{x^1}_{(1)})\alpha x^3{X^3}_{(2)} \\
&= {X^1}_{(1)}b_{(1)}S(X^2)_{(1)}g^1\gamma^1{X^3}_{(1)}\otimes{X^1}_{(2)}b_{(2)}S(X^2)_{(2)}g^2\gamma^2{X^3}_{(2)} \\
&= {X^1}_{(1)}b_{(1)}S(X^2)_{(1)}\alpha_{(1)}{X^3}_{(1)}\otimes{X^1}_{(2)}b_{(2)}S(X^2)_{(2)}\alpha_{(2)}{X^3}_{(2)} \\
&= \Delta(X^1bS(X^2)\alpha X^3) \\
&= \Delta(q^1bS(q^2))
\end{aligned}
$$

□

Example 3.2. Recall the structure of the twisted quantum double, $D^\phi(G)$, for a finite non-abelian group G from [3],

$$
\begin{gathered}
(g\otimes\delta_s)(h\otimes\delta_t) = (gh\otimes\delta_s)\delta_{s,gtg^{-1}}\ \ \theta_s(g,h) \\
\eta(1) = (e\otimes 1) \\
\Delta(g\otimes\delta_s) = \sum_{ab=s}(g\otimes\delta_a)\otimes(g\otimes\delta_b)\ \ \gamma_g(a,b) \\
\varepsilon(g\otimes\delta_s) = \delta_{s,e} \\
S(g\otimes\delta_s) = g^{-1}\otimes\delta_{s^{-1}}\ \ \theta^{-1}_{s^{-1}}(g,g^{-1})\gamma^{-1}_g(s,s^{-1})
\end{gathered}
$$

$$\alpha = (e \otimes 1)$$
$$\beta = \sum_g (e \otimes \delta_g)\ \ \phi(g^{-1}, g, g^{-1})$$
$$\phi_D = \sum_{g,h,k} (e \otimes \delta_g) \otimes (e \otimes \delta_h) \otimes (e \otimes \delta_k)\ \ \phi(g, h, k)$$
$$R = \sum_g (e \otimes \delta_g) \otimes (g \otimes 1)$$

where for all $g, h, t \in G$,

$$\theta_s(g, h) = \phi(g, g^{-1}sg, h)\phi^{-1}(s, g, h)\phi^{-1}(g, h, h^{-1}g^{-1}sgh)$$
$$\gamma_g(a, b) = \phi(a, g, g^{-1}bg)\phi^{-1}(a, b, g)\phi^{-1}(gg^{-1}ag, g^{-1}bg)$$

which further satisfy the following identities.

$$\theta_s(g, h)\theta_s(gh, k) = \theta_s(g, hk)\theta_{g^{-1}sg}(h, k)$$
$$\gamma_g(a, b)\gamma_g(ab, c)\phi(a, b, c) = \gamma_g(a, bc)\gamma_g(b, c)\phi(g^{-1}ag, g^{-1}bg, g^{-1}cg)$$
$$\theta_s(g, h)\theta_t(g, h)\gamma_g(s, t)\gamma_h(g^{-1}sg, s^{-1}tg) = \theta_{st}(g, h)\gamma_{gh}(s, t)$$

The adjoint action of $D^\phi(G)$ is given by

$$\begin{aligned}(g \otimes \delta_s) \triangleright (h \otimes \delta_t) = {} & (ghg^{-1} \otimes \delta_{gtg^{-1}})\, \delta_{s,gth^{-1}t^{-1}hg^{-1}} \\ & \gamma_g(gtg^{-1}, gt^{-1}g^{-1})\gamma_g^{-1}(gh^{-1}t^{-1}hg^{-1}, gh^{-1}thg^{-1}) \\ & \theta_{gtg^{-1}}(g, h)\theta_{gtg^{-1}}(gh, g^{-1})\theta^{-1}_{b^{-1}}(g, g^{-1})\end{aligned}$$

We note that $(e \otimes \delta_s) \triangleright (h \otimes \delta_t) = (h \otimes \delta_t)\, \delta_{s,th^{-1}t^{-1}h}$.

We find the structure of $\underline{D^\phi(G)}$ to be

$$\begin{aligned}\underline{m}((g \otimes \delta_s) \otimes (h \otimes \delta_t)) = {} & (gh \otimes \delta_s)\, \delta_{s,gtg^{-1}}\theta_s(g, h) \\ & \phi(s, g^{-1}s^{-1}g, g^{-1}sg)\phi^{-1}(sg^{-1}s^{-1}g, g^{-1}sg, h^{-1}g^{-1}s^{-1}gh)\end{aligned}$$

$$\underline{\eta}(1) = \sum_{g \in G} (e \otimes \delta_g)\, \phi(g^{-1}, g, g^{-1})$$

$$\begin{aligned}\underline{\Delta}(g \otimes \delta_s) = {} & \sum_{ab=s} (bgb^{-1} \otimes \delta_a) \otimes (g \otimes \delta_b)\, \gamma_g(a, b)\theta_a^{-1}(bgb^{-1}, bg^{-1}b^{-1}g)\phi(s, g^{-1}s^{-1}g, g^{-1}sg) \\ & \phi^{-1}(a, bg^{-1}b^{-1}a^{-1}bgb^{-1}, bg^{-1}b^{-1}abgb^{-1})\phi^{-1}(b, g^{-1}b^{-1}g, g^{-1}bg) \\ & \phi(bg^{-1}b^{-1}g, g^{-1}ag, g^{-1}bg)\phi^{-1}(abg^{-1}b^{-1}a^{-1}bgb^{-1}, bg^{-1}b^{-1}g, g^{-1}abg) \\ & \phi(abg^{-1}b^{-1}a^{-1}bgb^{-1}, bg^{-1}b^{-1}abgb^{-1}, b)\phi^{-1}(bg^{-1}b^{-1}abgb^{-1}, bg^{-1}b^{-1}g, g^{-1}bgb)\end{aligned}$$

$$\underline{\varepsilon}(g \otimes \delta_s) = \delta_{s,e}$$

$$\begin{aligned}\underline{S}(g \otimes \delta_s) = &(sg^{-1}s^{-1} \otimes \delta_{sg^{-1}s^{-1}gs^{-1}})\theta^{-1}_{s^{-1}}(g, g^{-1}\gamma_g^{-1}(s, s^{-1})\\ &\theta_{sg^{-1}s^{-1}gs^{-1}}(sg^{-1}s^{-1}g, g^{-1})\phi(sg^{-1}s^{-1}gs^{-1}, sg^{-1}s^{-1}g, g^{-1}sg)\\ &\phi(s, g^{-1}s^{-1}g, g^{-1}sg)\phi^{-1}(sg^{-1}s^{-1}g, g^{-1}s^{-1}g, g^{-1}sg)\phi(g^{-1}sg, g^{-1}s^{-1}g, g^{-1}sg)\end{aligned}$$

4. Bosonisation of Braided Groups in ${}_H\mathcal{M}$

Let H be a quasi-triangular quasi-Hopf algebra. Given a braided group in ${}_H\mathcal{M} = \mathcal{C}$ we can ‘ bosonise’ it back to an equivalent ordinary quasi-Hopf algebra. We use the same strategy as in [10]. If B is a braided Hopf algebra in $\mathcal{C}$, then a braided B-module is an object V in $\mathcal{C}$ and a morphism $\alpha_V^B : B \otimes V \to V$ in $\mathcal{C}$. Note that α_V^B intertwines the action of H, that is $\alpha_V^B(h\triangleright(b \otimes v)) = h\triangleright\alpha_V^B(b \otimes v)$, for all $h \in H, b \in B, v \in V$; equivalently,

$$h\triangleright(b\triangleright v) = (h_{(1)}\triangleright b)\triangleright(h_{(2)}\triangleright v)$$

where the notation for the actions of H on B, H on V and B on V are understood. The category ${}_B\mathcal{C}$ of braided B-modules in $\mathcal{C}$ is a braided monoidal category with the same braiding as $\mathcal{C}$.

Theorem 4.1. *Let H be a quasitriangular quasi-Hopf algebra, and $B \in \mathcal{C}$ be a braided group. Then there is an ordinary quasi-Hopf algebra $B \rtimes H$ built on the vector space $B \otimes H$ with structure*

$$\begin{gathered}(b \otimes h)(c \otimes g) = (x^1\triangleright b)\underline{\cdot}(x^2 h_{(1)}\triangleright c) \otimes x^3 h_{(2)} g\\ \eta(1) = 1_B \otimes 1\\ \Delta(b\otimes h) = y^1 X^1\triangleright b_{\underline{(1)}}\otimes y^2 Y^1 R^{(2)} x^2 X^3{}_{(1)} h_{(1)}\otimes y^3{}_{(1)} Y^2 R^{(1)} x^1 X^2\triangleright b_{\underline{(2)}}\otimes y^3{}_{(2)} Y^3 x^3 X^3{}_{(2)} h_{(2)}\\ \varepsilon(b \otimes h) = \underline{\varepsilon}(b)\varepsilon(h)\\ S(b\otimes h) = (S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(1)} X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b)\otimes(S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(2)} X^3 x^2 \beta S(x^3)\\ \alpha_{B\rtimes H} = 1_B \otimes \alpha\\ \beta_{B\rtimes H} = 1_B \otimes \beta\\ \phi_{B\rtimes H} = 1_B \otimes X^1 \otimes 1_B \otimes X^2 \otimes 1_B \otimes X^3\end{gathered}$$

Proof. Given a braided B-module, V, in $\mathcal{C}$, we have an action of B on V and an action of H on V. The action of $B \rtimes H$ on V is

$$(b \otimes h)\triangleright v = b\triangleright(h\triangleright v)$$

for all $v \in V, b \in B, h \in H$. Note, that since the action of B on V is a morphism in $\mathcal{C}$ it satisfies

$$b\triangleright(c\triangleright v) = (x^1\triangleright b)(x^2\triangleright c)\triangleright(x^3\triangleright v)$$

for all $b, c \in B, v \in V$. Since $B \rtimes H$ is an ordinary Hopf algebra, it satisfies

$$(b\otimes h)(c\otimes g)\triangleright v=(b\otimes h)\triangleright((c\otimes g)\triangleright v)$$

and hence this determines the multiplication in $B \rtimes H$.

$$\begin{aligned}&(b\otimes h)(c\otimes g)\triangleright v=b\triangleright(h\triangleright((c\otimes g)\triangleright v))\\&=b\triangleright(h\triangleright(c\triangleright(g\triangleright v)))\\&=b\triangleright((h_{(1)}\triangleright c)\triangleright(h_{(2)}\triangleright(g\triangleright v)))\\&=b\triangleright((h_{(1)}\triangleright c)\triangleright(h_{(2)}g\triangleright v))\\&=(x^1\triangleright b)(x^2\triangleright(h_{(1)}\triangleright c))\triangleright(x^3\triangleright(h_{(2)}g\triangleright v))\\&=(x^1\triangleright b)(x^2h_{(1)}\triangleright c)\triangleright(x^3h_{(2)}g\triangleright v)\end{aligned}$$

Thus,

$$(b\otimes h)(c\otimes g)=(x^1\triangleright b)(x^2h_{(1)}\triangleright c)\otimes x^3h_{(2)}g.$$

Let $V,W\in{}_B\mathcal{C}$then $B \rtimes H$ acts on $V\otimes W$ as

$$(b\otimes h)\triangleright(v\otimes w)=(b\otimes h)_{(1)}\triangleright v\otimes(b\otimes h)_{(2)}\triangleright w$$

for all $v\in V, w\in W$. But also

$$(b\otimes h)\triangleright(v\otimes w)=b\triangleright(h\triangleright(v\otimes w))=b\triangleright(h_{(1)}\triangleright v\otimes h_{(2)}\triangleright w)$$

Thus the coproduct of $B \rtimes H$ is characterised by

$$(b\otimes h)_{(1)}\triangleright v\otimes(b\otimes h)_{(2)}\triangleright w=b\triangleright(h_{(1)}\triangleright v\otimes h_{(2)}\triangleright w)$$

Now, B acts on the tensor product $V\otimes W$ as

$$\alpha^B_{V\otimes W}=(\alpha^B_V\otimes\alpha^B_W)\Phi^{-1}_{B,V,B\otimes W}(\mathrm{id}\otimes\Phi_{V,B,W})(\mathrm{id}\otimes\Psi_{B,V}\otimes\mathrm{id})(\mathrm{id}\otimes\Phi^{-1}_{B,V,W})\Phi_{B,B,V\otimes W}(\underline{\Delta}\otimes\mathrm{id}\otimes\mathrm{id})$$

that is,

$$b\triangleright(v\otimes w)=(y^1X^1\triangleright b_{\underline{(1)}})\triangleright(y^2Y^1R^{(2)}x^2X^3{}_{(1)}\triangleright v)\otimes(y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright b_{\underline{(2)}})\triangleright(y^3{}_{(2)}Y^3x^3X^3{}_{(2)}\triangleright w)$$

So,

$$\begin{aligned}&(b\otimes h)\triangleright(v\otimes w)=b\triangleright(h_{(1)}\triangleright v\otimes h_{(2)}\triangleright w)\\&=(y^1X^1\triangleright b_{\underline{(1)}})\triangleright(y^2Y^1R^{(2)}x^2X^3{}_{(1)}h_{(1)}\triangleright v)\otimes(y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright b_{\underline{(2)}})\triangleright(y^3{}_{(2)}Y^3x^3X^3{}_{(2)}h_{(2)}\triangleright w)\end{aligned}$$

Thus,

$$\Delta(b\otimes h)=y^1X^1\triangleright\underline{b_{(1)}}\otimes y^2Y^1R^{(2)}x^2X^3{}_{(1)}h_{(1)}\otimes y^3{}_{(1)}Y^2R^{(1)}x^1X^2\triangleright\underline{b_{(2)}}\otimes y^3{}_{(2)}Y^3x^3X^3{}_{(2)}h_{(2)}$$

For the antipode, given $V\in {}_B\mathcal{C}$ we have to consider how H and B act on the dual object V^*. It is known that for a quasi-Hopf algebra H and a left H-module V, the dual space V^* becomes a left H-module by $(h\triangleright v^*)(v)=v^*(h\triangleright v)$ for any $v^*\in V^*, v\in V, h\in H$, thus for $B\rtimes H$,

$$((b\otimes h)\triangleright v^*)(v)=v^*(S(b\otimes h)\triangleright v)$$

for all $v^*\in V^*, v\in V, b\in B, h\in H$. But we also have

$$((b\otimes h)\triangleright v^*)(v)=(b\triangleright(h\triangleright v^*))(v)$$

so the antipode is determined by

$$v^*(S(b\otimes h)\triangleright v)=(b\triangleright(h\triangleright v^*))(v)$$

so it remains to find how B acts on the dual space. If V is a left B-module, then V^* is a right B-module by $\alpha^*: V^*\otimes B\to V^*$ as

$$\alpha^*=l_{V^*}^{-1}(ev_V\otimes\mathrm{id})(\mathrm{id}\otimes\alpha_V^B\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes coev_V)(\mathrm{id}\otimes r_B)$$

so,

$$\begin{aligned}
&(\alpha^*(v^*\otimes b))(v)=(l_{V^*}^{-1}\dots(\mathrm{id}\otimes r_B)(v^*\otimes b))(v)\\
&=l_{V^*}^{-1}\dots\Phi^{-1}_{V^*,B,\underline{1}}(v^*\otimes(b\otimes 1))(v)\\
&=l_{V^*}^{-1}\dots(\mathrm{id}\otimes\mathrm{id}\otimes coev_V)((x^1\triangleright v^*\otimes x^2\triangleright b)\otimes x^3\triangleright 1)(v)\\
&=l_{V^*}^{-1}\dots(\mathrm{id}\otimes\mathrm{id}\otimes coev_V)((x^1\triangleright v^*\otimes x^2\triangleright b)\otimes\varepsilon(x^3))(v)\\
&=l_{V^*}^{-1}\dots(\mathrm{id}\otimes\mathrm{id}\otimes coev_V)((v^*\otimes b)\otimes 1)(v)\\
&=l_{V^*}^{-1}\dots\Phi^{-1}_{V^*\otimes B,V,V^*}((v^*\otimes b)\otimes(\beta\triangleright e_a\otimes f^a))(v)\\
&=l_{V^*}^{-1}\dots(\Phi_{V^*,B,V}\otimes\mathrm{id})(((x^1{}_{(1)}\triangleright v^*\otimes x^1{}_{(2)}\triangleright b)\otimes x^3\beta\triangleright e_a)\otimes x^3\triangleright f^a)(v)\\
&=l_{V^*}^{-1}\dots(\mathrm{id}\otimes\alpha_V^B\otimes\mathrm{id})((X^1x^1{}_{(1)}\triangleright v^*\otimes(X^2x^1{}_{(2)}\triangleright b\otimes X^3x^2\beta\triangleright e_a))\otimes x^3\triangleright f^a)(v)\\
&=l_{V^*}^{-1}(ev_V\otimes\mathrm{id})((X^1x^1{}_{(1)}\triangleright v^*\otimes(X^2x^1{}_{(2)}\triangleright b)\triangleright(X^3x^2\beta\triangleright e_a))\otimes x^3\triangleright f^a)(v)\\
&=l_{V^*}^{-1}((X^1x^1{}_{(1)}\triangleright v^*)(\alpha\triangleright((X^2x^1{}_{(2)}\triangleright b)\triangleright(X^3x^2\beta\triangleright e_a)))\otimes x^3\triangleright f^a)(v)\\
&=l_{V^*}^{-1}(v^*(S(X^1x^1{}_{(1)})\alpha\triangleright((X^2x^1{}_{(2)}\triangleright b)\triangleright(X^3x^2\beta\triangleright e_a)))\otimes x^3\triangleright f^a)(v)\\
&=v^*(S(X^1x^1{}_{(1)})\alpha\triangleright((X^2x^1{}_{(2)}\triangleright b)\triangleright(X^3x^2\beta S(x^3)\triangleright v)))
\end{aligned}$$

Then, V^* becomes a left B-module by

$$\begin{aligned}
&\alpha^B_{V^*}(b\otimes v^*)(v)=\alpha^*(\mathrm{id}\otimes\underline{S})\Psi_{B,V^*}(b\otimes v^*)(v)\\
&=\alpha^*(\mathrm{id}\otimes\underline{S})(R^{(2)}\triangleright v^*\otimes R^{(1)}\triangleright b)(v)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha^*(R^{(2)}\triangleright v^* \otimes \underline{S}(R^{(1)}\triangleright b))(v)\\
&= \alpha^*(R^{(2)}\triangleright v^* \otimes R^{(1)}\triangleright \underline{S}(b))(v)\\
&= (R^{(2)}\triangleright v^*)(S(X^1 x^1{}_{(1)})\alpha\triangleright((X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright(X^3 x^2 \beta S(x^3)\triangleright v)))\\
&= v^*(S(R^{(2)})S(X^1 x^1{}_{(1)})\alpha\triangleright((X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright(X^3 x^2 \beta S(x^3)\triangleright v)))
\end{aligned}
$$

So, the action of $B \rtimes H$ on V^* is given by

$$
\begin{aligned}
&((b\otimes h)\triangleright v^*)(v) = (b\triangleright(h\triangleright v^*))(v)\\
&= (h\triangleright v^*)(S(R^{(2)})S(X^1 x^1{}_{(1)})\alpha\triangleright((X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright(X^3 x^2 \beta S(x^3)\triangleright v)))\\
&= v^*(S(h)S(R^{(2)})S(X^1 x^1{}_{(1)})\alpha\triangleright((X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright(X^3 x^2 \beta S(x^3)\triangleright v)))\\
&= v^*(S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha\triangleright((X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright(X^3 x^2 \beta S(x^3)\triangleright v)))\\
&= v^*(((S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(1)} X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright((S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(2)} X^3 x^2 \beta S(x^3)\triangleright v))
\end{aligned}
$$

So

$$
\begin{aligned}
&((b\otimes h)\triangleright v^*)(v) = v^*(S(b\otimes h)\triangleright v)\\
&= v^*(((S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(1)} X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b))\triangleright((S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(2)} X^3 x^2 \beta S(x^3)\triangleright v))
\end{aligned}
$$

Hence,

$$
S(b\otimes h) = (S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(1)} X^2 x^1{}_{(2)} R^{(1)}\triangleright\underline{S}(b)\otimes(S(X^1 x^1{}_{(1)} R^{(2)} h)\alpha)_{(2)} X^3 x^2 \beta S(x^3).
$$

□

Corollary 4.2. *The modules of B in the braided category ${}_H\mathcal{M}$ correspond to the ordinary modules of $B \rtimes H$. The braided categories ar isomorphic.*

Proof. $B \rtimes H$ is a smash product when considered as an algebra. This structure was found in [1], and this correspondence is given as follows. Let V be a $B \rtimes H$-module with structure given by $(b\otimes h)\cdot v$. Define maps $j : H \to B \rtimes H$ and $i : B \to B \rtimes H$ by $j(h) = 1 \otimes h$ and $i(b) = b \otimes 1$. Then V becomes a left H-module by $h\triangleright v = j(h)\cdot v$, and V becomes a braided B-module by $b\triangleright v = i(b)\cdot v$. Conversely, if V is a braided module in ${}_H\mathcal{M}$, define the action of $B \rtimes H$ on V by $(b \otimes h)\cdot v = b\triangleright(h\triangleright v)$. It is straightforward to see that this is an equivalence of monoidal categories by the same steps as in [10].

□

Example 4.3. For an example of braided group bosonisation, we consider the group function algebra $k_\phi(G)$, and find an isomorphism $\underline{kG} \rtimes k_\phi(G) \cong D^\phi(G)$.

Consider the group function algebra, $k(G)$, of a finite group G with identity e. This is the set of functions on G with values in k. This has the structure of a commutative Hopf algebra as follows.

$$\begin{aligned}\delta_s\cdot\delta_t &= \delta_t\ \ \delta_{s,t}\\ \eta(1) &= \sum_{t\in G}\delta_t\\ \Delta(\delta_t) &= \sum_{ab=t}\delta_a\otimes\delta_b\\ \varepsilon(\delta_t) &= \delta_{t,e}\\ S(\delta_t) &= \delta_{t^{-1}}\end{aligned}$$

for all $\delta_s,\delta_t\in k(G)$. Further, one can view $k(G)$ as a quasi-Hopf algebra with $\phi_G\in k(G)^{\otimes 3}$ defined by

$$\phi_G=\sum_{r,s,t\in G}\delta_r\otimes\delta_s\otimes\delta_t\ \ \phi(r,s,t)$$

for some 3-cocycle $\phi\in k(G)$ satisfying

$$\begin{aligned}\phi(b,c,d)\phi(a,bc,d)\phi(a,b,c) &= \phi(a,b,cd)\phi(ab,c,d)\\ \phi(a,e,b) &= 1\end{aligned}$$

for all $a,b,c,d\in G$. Choosing $\alpha=\varepsilon_{kG}=1$, this determines $\beta\in k(G)$ as

$$\beta=\sum_{t\in G}\delta_t\ \ \phi^{-1}(t,t^{-1},t)=\sum_{t\in G}\delta_t\ \ \phi(t^{-1},t,t^{-1})$$

A quasitriangular structure for $k(G)$ as a quasi-Hopf algebra is defined by $R=\sum_{s,t\in G}\delta_s\otimes\delta_t\ \ r(s,t)$, where $r\in k(G)\otimes k(G)$ is a function obeying

$$\begin{aligned}r(gh,t) &= r(g,t)r(h,t)\frac{\phi(t,g,h)\phi(g,h,t)}{\phi(g,t,h)}\\ r(t,gh) &= r(t,g)r(t,h)\frac{\phi(g,t,h)}{\phi(t,g,h)\phi(g,h,t)}\\ r(u,e) &= 1=r(e,u)\end{aligned}$$

for all $g,h,t\in G$. We denote this quasitriangular quasi-Hopf algebra by $k_\phi(G)$. The structure of $\underline{k_\phi(G)}$ is as follows

$$\underline{m}(\delta_s\otimes\delta_t)=\delta_t\,\delta_{s,t}\phi(t,t^{-1},t)$$

$$\underline{\eta}(1)=\sum_{s\in G}\delta_s\ \ \phi(s^{-1},s,s^{-1})$$

$$\underline{\Delta}(\delta_t)=\sum_{ab=t}\delta_a\otimes\delta_b\ \ \frac{\phi(t,t^{-1},t)}{\phi(a,a^{-1},a)\phi(b,b^{-1},b)}$$

$$\underline{\varepsilon}(\delta_s)=\delta_{s,e}$$

$$\underline{S}(\delta_s) = \delta_{s^{-1}} \ \ \phi(s, s^{-1}, s)\phi(s, s^{-1}, s)$$

So $\underline{k(G)}$ has structure

$$\begin{aligned}
\delta_s \underline{\cdot} \delta_t &= \delta_t \ \ \delta_{s,t}\phi(t, t^{-1}, t) \\
\underline{\eta}(1) &= \sum_{t \in G} \delta_t \phi(t^{-1}, t, t^{-1}) \\
\underline{\Delta}\delta_t &= \sum_{ab=t} \delta_a \otimes \delta_b \ \ \frac{\phi(t, t^{-1}, t)}{\phi(a, a^{-1}, a)\phi(b, b^{-1}, b)} \\
\underline{\varepsilon}(\delta_t) &= \delta_{t,e} \\
\underline{S}(\delta_t) &= \delta_{t^{-1}} \ \ \phi(t, t^{-1}, t)\phi(t, t^{-1}, t)
\end{aligned}$$

We can find the structure on $\underline{kG} = \underline{(k_\phi(G))^*} = (\underline{k_\phi(G)})^*$ as this braided dual structure is determined by the structure on the original braided group. First, consider how $\underline{kG} \in {}_{k_\phi(G)}\mathcal{M}$. Let $\psi \in k_\phi(G), \delta_s \in \underline{k_\phi(G)}, g \in \underline{kG}$, then

$$\langle \psi \triangleright g, \delta_s \rangle = \langle g, S(\psi) \triangleright \delta_s \rangle$$

So,

$$\begin{aligned}
\delta_s(\psi \triangleright g) &= (S(\psi) \triangleright \delta_s)(g) \\
&= ((S\psi)_{(1)} \delta_s S((S\psi)_{(2)}))(g) \\
&= (S\psi)_{(1)}(g) \delta_s(g) (S\psi)_{(2)}(g^{-1}) \\
&= (S\psi)(e)\delta_s(g) \\
&= \psi(e)\delta_s(g) \\
&= \delta_s(\psi(e)g)
\end{aligned}$$

Thus, $\psi \triangleright g = \psi(e)g$ for all $\psi \in k_\phi(G), g \in \underline{kG}$. So, if we consider the associativity constraint Φ on this category; if it is acting on $\underline{kG}$, it is in fact trivial, and as such, in the following calculations we can ignore the bracketing order. Now, the multiplication on $\underline{kG}$ is determined by the comultiplication on $\underline{k_\phi(G)}$ as follows:

$$ev(r^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes ev \otimes \mathrm{id})(\underline{\Delta} \otimes \mathrm{id} \otimes \mathrm{id}) = ev(\mathrm{id} \otimes \underline{m}) : \underline{k(G)} \otimes \underline{kG} \otimes \underline{kG} \to \underline{1}$$

The left hand side gives

$$\begin{aligned}
&ev(r^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes ev \otimes \mathrm{id})(\underline{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(\delta_s \otimes g \otimes h) \\
&= ev(r^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes ev \otimes \mathrm{id})((\delta_s)_{\underline{(1)}} \otimes (\delta_s)_{\underline{(2)}} \otimes g \otimes h) \\
&= ev((\delta_s)_{\underline{(1)}} \otimes h \, (\delta_s)_{\underline{(2)}}(\alpha \triangleright g)) \\
&= (\delta_s)_{\underline{(1)}}(\alpha \triangleright h)(\delta_s)_{\underline{(2)}}(\alpha \triangleright g) \\
&= \underline{\Delta}(\delta_s)(h, g)
\end{aligned}$$

$$= \frac{\phi(hg,(hg)^{-1},hg)}{\phi(h,h^{-1},h)\phi(g,g^{-1},g)}\delta_s(hg)$$
$$= \frac{\phi(gh,(gh)^{-1},gh)}{\phi(g,g^{-1},g)\phi(h,h^{-1},h)}\delta_s(gh)$$

while the right hand side gives

$$\begin{aligned} ev(\mathrm{id}\otimes \underline{m})(\delta_s\otimes g\otimes h) &= ev(\delta_s\otimes g\underline{\cdot} h)\\ &= \delta_s(\alpha\triangleright(g\underline{\cdot} h))\\ &= \delta_s(g\underline{\cdot} h)\end{aligned}$$

These are equal, hence $\underline{m}(g\otimes h) = \frac{\phi(gh,(gh)^{-1},gh)}{\phi(g,g^{-1},g)\phi(h,h^{-1},h)}\, gh$ for all $g,h\in \underline{kG}$. The rest of the structure is similarly determined; the structure of $\underline{kG}$ is

$$\begin{aligned} \underline{m}(g\otimes h) &= gh\ \ \frac{\phi(gh,(gh)^{-1},gh)}{\phi(g,g^{-1},g)\phi(h,h^{-1},h)}\\ \underline{\eta}(1) &= e\\ \underline{\Delta}(g) &= g\otimes g\ \ \phi(g,g^{-1},g)\\ \underline{\varepsilon}(g) &= \phi(g^{-1},g,g^{-1})\\ \underline{S}(g) &= g^{-1}\ \ \phi(g^{-1},g,g^{-1})\phi(g^{-1},g,g^{-1})\end{aligned}$$

Finally, we can bosonise $\underline{kG}\in {}_{k_\phi(G)}\mathcal{M}$ into an ordinary quasi-Hopf algebra with the following structure:

$$\begin{aligned} (g\otimes\delta_s)(h\otimes\delta_t) &= gh\otimes\delta_{s,t}\delta_t\frac{\phi(gh,(gh)^{-1},gh)}{\phi(g,g^{-1},g)\phi(h,h^{-1},h)}\\ \eta(1)(g\otimes\delta_t) &= e\otimes 1\\ \Delta(g\otimes\delta_t) &= \sum_{ab=t} g\otimes\delta_a\otimes g\otimes\delta_b\,\phi(g,g^{-1},g)\\ \varepsilon(g\otimes\delta_t) &= \phi(g^{-1},g,g^{-1})\delta_{t,e}\\ S(g\otimes\delta_t) &= g^{-1}\otimes\delta_{t^{-1}}\phi(g^{-1},g,g^{-1})\phi(g^{-1},g,g^{-1})\end{aligned}$$

There exists a quasi-Hopf algebra isomorphism $\sigma:\underline{kG}\rtimes k_\phi(G)\to D^\phi(G)$defined by

$$\sigma(g\otimes\delta_t) = g\otimes\delta_t\ \ \phi(g^{-1},g,g^{-1})R^{-1}(g,t)$$

It is straightforward to check that σ is an isomorphism of quasi-Hopf algebras. Using this isomorphism and its inverse, one can obtain the quasitriangular structure of $\underline{kG}\rtimes k_\phi(G)$. Note that $\sigma^{-1}(g\otimes\delta_t) = g\otimes\delta_t\ \ \phi^{-1}(g^{-1},g,g^{-1})R(g,t) = g\otimes\delta_t\ \ \phi(g,g^{-1},g)R(g,t)$, hence,

$$\begin{aligned} R_B &= (\sigma^{-1}\otimes\sigma^{-1})(R_D)\\ &= \sum_{g\in G}(\sigma^{-1}\otimes\sigma^{-1})(e\otimes\delta_g\otimes g\otimes 1)\end{aligned}$$

$$= \sum_{g,h \in G} e \otimes \delta_g \otimes g \otimes 1 \;\; \phi(g, g^{-1}, g) R(g,h)$$
$$= \sum_{g \in G} e \otimes \delta_g \otimes g \otimes 1 \;\; \phi(g, g^{-1}, g) R(g,g)$$

Remark 4.4. Similarly, we would expect $\underline{H}^* \rtimes H \cong D^{\phi}(H)$ for any quasi-Hopf algebra H. When H is factorisable, $\underline{H}^* \cong \underline{H}$, and this case is covered in the next section.

Example 4.5. Following [8], we consider a group G and an invetible 2-cochain $F : G \times G \to k^*$ satisfying $F(e,g) = F(g,e) = 1$ for all $g \in G$. Then one can consider the deformation of the group algebra kG with modified product

$$g \cdot_F h = F(g,h) gh$$

for all $g, h \in G$ and where gh is the usual group product in G.

For a group G and $\phi : G \times G \times G \to k^*$ an invertible group 3-cocycle, the category of G-graded vector spaces (the category of $k(G)$-modules is monoidal with associator determined by ϕ and the grading. From [7], $k_F G$ is a G-graded quasialgebra with $|g| = g$ for $g \in G$, which is quasiassociative with associator ϕ the coboundary of F, that is,

$$(g \cdot_F h) \cdot_F k = \phi(|g|, |h|, |k|) g \cdot_F (h \cdot_F k)$$
$$\phi(g,h,k) = \frac{F(g,h)F(gh,k)}{F(h,k)F(g,hk)}$$

for all $g, h, k \in G$. Here $\phi : G \times G \times G \to k^*$ is an invertible 3-cocycle, and gives the category of G-graded vector spaces a monoidal structure.

If G is abelian and ϕ is of coboundary form ($\phi = \partial F$), the category of G-graded spaces is braided with Ψ determined by the function $R(g,h) = \frac{F(g,h)}{F(h,g)}$ and kG_F is quasicommutative with $g \cdot_F h = R(g,h) h \cdot_F g$.

In the case when $G = \mathbb{Z}_2^n$, F takes the form $F(g,h) = (-1)^{f(g,h)}$ where f is a $\mathbb{Z}_2$-valued function on $G \times G$ such that $F^2 = 1$. The octonions are of this form for the group group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with

$$f(g,h) = \sum_{i \le j} g_i h_j + h_1 g_2 g_3 + g_1 h_2 g_3 + g_1 g_2 h_3$$
$$\phi(g,h,k) = (-1)^{(g \times h) \cdot k} = (-1)^{|ghk|}$$
$$R(g,h) = \begin{cases} 1 \text{ if } g = e \text{ or } h = e \text{ or } g = h \\ -1 \text{ otherwise} \end{cases}$$

We let $G = \mathbb{Z}_2^3$, and consider the graded basis $\{e_a | a \in \mathbb{Z}_2\}$ of kG_F and the dual basis of delta functions $\{\delta_a\}$ of the group function algebra $k(G)$. We can view a G-graded quasialgebra as a $k_\phi(G)$-module quasi-algebra with action $\delta_b \triangleright e_a = \delta_b(|e_a|) e_a = \delta_{b,a} e_a$ on homogeneous elements, where $k_\phi(G)$ is the usual group function algebra on G regarded as a quasi-Hopf algebra with $\phi \in k(G)^{\otimes 3}$. Thus, the algebra of

octonions,kG_F, live naturally in the category of $k_\phi(G)$-modules, and as such we can construct the bosonisation of the octonions as an algebra.

$$\begin{aligned}
(e_a \otimes \delta_s)(e_b \otimes \delta_t) &= (\phi^{-(1)} \triangleright e_a) \cdot_F (\phi^{-(2)}(\delta_s)_{(1)} \triangleright e_b) \otimes \phi^{-(3)}(\delta_s)_{(2)} \delta_t \\
&= \sum_{xy=s} (\phi^{-(1)} \triangleright e_a) \cdot_F (\phi^{-(2)} \delta_x \triangleright e_b) \otimes \phi^{-(3)} \delta_y \delta_t \\
&= \sum_{xy=s} \phi^{-(1)}(|e_a|)\phi^{-(2)}(|e_b|)\delta_a(|e_b|)\, e_a \cdot_F e_b \otimes \phi^{-(3)} \delta_y \delta_t \\
&= \sum_{xy=s} e_a \cdot_F e_b \otimes \phi^{-(1)}(a)\phi^{-(2)}(b)\delta_x(b)\phi^{-(3)}\delta_b \delta_t \\
&= e_a \cdot_F e_b \otimes \delta_t \delta_{-b+s,t} \phi^{-1}(a,b,t) \\
&= (-1)^{|abt|}\, e_a \cdot_F e_b \otimes \delta_{-b+s,t} \delta_t
\end{aligned}$$

for all $a, b, s, t \in \mathbb{Z}_2^3$.

It is clear that $(1 \otimes \delta_s)(1 \otimes \delta_t) = 1 \otimes \delta_s \delta_t$, and so $\mathbb{O} \rtimes k_\phi(\mathbb{Z}_2^3) \supset k_\phi(\mathbb{Z}_2^3)$ as a subalgebra. It also contains an algebra with the following structure.

$$\begin{aligned}
(e_a \otimes 1)(e_b \otimes 1) &= \sum_{s,t} (-1)^{|abt|} e_a \cdot_F e_b \otimes \delta_{-b+s,t} \\
&= \sum_{t} (-1)^{|abt|} e_a \cdot_F e_b \otimes \delta_t \\
&= e_a \cdot_F e_b \chi(a,b)
\end{aligned}$$

where $\chi(a,b) = \sum_t (-1)^{|abt|} \delta_t$, and we note

$$\chi(a,b) = \begin{cases} 1 & \text{if } a = 0 \text{ or } b = 0 \text{ or } a = b \\ 2(\delta_0 + \delta_a + \delta_b + \delta_{a+b}) - 1 & \text{otherwise} \end{cases}$$

Finally, the commutation relations are

$$\begin{aligned}
(e_a \otimes 1)(1 \otimes \delta_t) &= \sum_s e_a \otimes \delta_{s,t} \delta_t \\
&= e_a \otimes \delta_t
\end{aligned}$$

$$\begin{aligned}
(1 \otimes \delta_s)(e_b \otimes 1) &= \sum_t e_b \otimes \delta_{s-b,t} \delta_t \\
&= e_b \otimes \delta_{s-b}
\end{aligned}$$

So we find that $f e_a = e_a L_a(f)$ for all $f \in k_\phi(\mathbb{Z}_2^3)$ and $a \in \mathbb{Z}_2^3$, where $L_a(f)(s) = f(a+s)$.

5. An Isomorphism $\underline{H} \rtimes H \cong H_{\mathcal{R}} \bowtie H$

Let $H_{\mathcal{R}} \bowtie H$ be the quasi-Hopf algebra with tensor product algebra, and coproduct

$$\begin{aligned}\Delta(b \otimes h) = {} & x^1Y^1b_{(1)}y^1X^1 \otimes x^2T^1R^{(2)}w^2{Y^3}_{(1)}h_{(1)}{y^3}_{(1)}W^2R^{-(2)}t^1X^2 \\ & \otimes {x^3}_{(1)}T^2R^{(1)}w^1Y^2b_{(2)}y^2W^1R^{-(1)}t^2{X^3}_{(1)} \otimes {x^3}_{(2)}T^3w^3{Y^3}_{(2)}h_{(2)}{y^3}_{(2)}W^3t^3{X^3}_{(2)}\end{aligned}$$

Theorem 5.1. *Let H be a quasitriangular quasi-Hopf algebra. There is a quasi-Hopf algebra isomorphism $\underline{H} \rtimes H \cong H_{\mathcal{R}} \bowtie H$ defined by*

$$\chi(a \otimes h) = q^1(x^1 \triangleright a)S(q^2)x^2h_{(1)} \otimes x^3h_{(2)}$$

Proof. It is straightforward to check that the inverse map is $\chi^{-1}(a \otimes h) = x^1aX^1\beta S(x^2h_{(1)}X^2) \otimes x^3h_{(2)}X^3$. First we show that χ is an algebra morphism,

$$\begin{aligned}\chi((a \otimes h)(b \otimes g)) &= \chi(q^1(y^1x^1 \triangleright a)S(q^2)y^2(x^2h_{(1)} \triangleright b)S(y^3) \otimes x^3h_{(2)}g) \\
&= Q^1(w^1 \triangleright (q^1(y^1x^1 \triangleright a)S(q^2)y^2(x^2h_{(1)} \triangleright b)S(y^3)))S(Q^2)w^2{x^3}_{(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3{x^3}_{(1)}h_{(2)(2)}g_{(2)} \\
&= \underline{X^1{w^1}_{(1)}}Y^1{y^1}_{(1)}{x^1}_{(1)}aS(Y^2{y^1}_{(2)}{x^1}_{(2)})\alpha Y^3y^2{x^2}_{(1)}h_{(1)(1)}b \\
&\quad S(\underline{X^2{w^1}_{(2)}}y^3{x^2}_{(2)}h_{(1)(2)})\alpha\underline{X^3w^2}{x^3}_{(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes \underline{w^3}{x^3}_{(2)}h_{(2)(2)}g_{(2)} \\
&= X^1\underline{Y^1{y^1}_{(1)}}{x^1}_{(1)}aS(\underline{Y^2{y^1}_{(2)}}{x^1}_{(2)})\alpha\underline{Y^3y^2}{x^2}_{(1)}h_{(1)(1)}b \\
&\quad S(w^1X^2\underline{y^3}{x^2}_{(2)}h_{(1)(2)})\alpha w^2{X^3}_{(1)}{x^3}_{(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3{X^3}_{(2)}{x^3}_{(2)}h_{(2)(2)}g_{(2)} \\
&= X^1\underline{Y^1{x^1}_{(1)}}aS(y^1\underline{Y^2{x^1}_{(2)}})\alpha y^2\underline{{Y^3}_{(1)}{x^2}_{(1)}}h_{(1)(1)}b \\
&\quad S(w^1X^2y^3\underline{{Y^3}_{(2)}{x^2}_{(2)}}h_{(1)(2)})\alpha w^2{X^3}_{(1)}\underline{{x^3}_{(1)}}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3{X^3}_{(2)}\underline{{x^3}_{(2)}}h_{(2)(2)}g_{(2)} \\
&= X^1t^1Y^1aS(\underline{y^1{t^2}_{(1)}}x^1Y^2)\alpha\underline{y^2{t^2}_{(2)(1)}}{x^2}_{(1)}{Y^3}_{(1)(1)}h_{(1)(1)}b \\
&\quad S(w^1X^2\underline{y^3{t^2}_{(2)(2)}}{x^2}_{(2)}{Y^3}_{(1)(2)}h_{(1)(2)})\alpha w^2{X^3}_{(1)}{t^3}_{(1)}{x^3}_{(1)} \\
&\quad {Y^3}_{(2)(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3{X^3}_{(2)}{t^3}_{(2)}{x^3}_{(2)}{Y^3}_{(2)(2)}h_{(2)(2)}g_{(2)} \\
&= \underline{X^1t^1}Y^1aS(y^1x^1Y^2)\alpha y^2{x^2}_{(1)}{Y^3}_{(1)(1)}h_{(1)(1)}b \\
&\quad S(w^1\underline{X^2t^2}y^3{x^2}_{(2)}{Y^3}_{(1)(2)}h_{(1)(2)})\alpha w^2\underline{{X^3}_{(1)}{t^3}_{(1)}}{x^3}_{(1)} \\
&\quad {Y^3}_{(2)(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3\underline{{X^3}_{(2)}{t^3}_{(2)}}{x^3}_{(2)}{Y^3}_{(2)(2)}h_{(2)(2)}g_{(2)} \\
&= Y^1aS(\underline{y^1x^1}Y^2)\alpha\underline{y^2{x^2}_{(1)}}{Y^3}_{(1)(1)}h_{(1)(1)}b \\
&\quad S(w^1\underline{y^3{x^2}_{(2)}}{Y^3}_{(1)(2)}h_{(1)(2)})\alpha w^2\underline{{x^3}_{(1)}} \\
&\quad {Y^3}_{(2)(1)}h_{(2)(1)}g_{(1)} \\
&\quad \otimes w^3\underline{{x^3}_{(2)}}{Y^3}_{(2)(2)}h_{(2)(2)}g_{(2)}\end{aligned}$$

$$
\begin{aligned}
&= Y^1 aS(x^1Y^2)\alpha x^2 \underline{X^1 Y^3{}_{(1)(1)} h_{(1)(1)}} bS(w^1 x^3{}_{(1)} \underline{X^2 Y^3{}_{(1)(2)} h_{(1)(2)}})\\
&\quad \alpha w^2 x^3{}_{(2)(1)} \underline{X^3{}_{(1)} Y^3{}_{(2)(1)} h_{(2)(1)}} g_{(1)}\\
&\quad \otimes w^3 x^3{}_{(2)(2)} \underline{X^3{}_{(2)} Y^3{}_{(2)(2)} h_{(2)(2)}} g_{(2)}\\
&= Y^1 aS(x^1Y^2)\alpha x^2 Y^3{}_{(1)} h_{(1)} X^1 bS(\underline{w^1 x^3{}_{(1)} Y^3{}_{(2)(1)} h_{(2)(1)}} X^2)\\
&\quad \alpha \underline{w^2 x^3{}_{(2)(1)} Y^3{}_{(2)(2)(1)} h_{(2)(2)(1)}} X^3{}_{(1)} g_{(1)}\\
&\quad \otimes \underline{w^3 x^3{}_{(2)(2)} Y^3{}_{(2)(2)(2)} h_{(2)(2)(2)}} X^3{}_{(2)} g_{(2)}\\
&= \underline{Y^1} aS(\underline{x^1 Y^2})\alpha \underline{x^2 Y^3{}_{(1)}} h_{(1)} X^1 bS(w^1 X^2)\alpha w^2 X^3{}_{(1)} g_{(1)}\\
&\quad \otimes \underline{x^3 Y^3{}_{(2)}} h^3{}_{(2)} w^3 X^3{}_{(2)} g_{(2)}\\
&= Y^1 x^1{}_{(1)} aS(Y^2 x^1{}_{(2)})\alpha Y^3 x^2 h_{(1)} \underline{X^1} bS(\underline{w^1 X^2})\alpha \underline{w^2 X^3{}_{(1)}} g_{(1)}\\
&\quad \otimes x^3 h^3{}_{(2)} \underline{w^3 X^3{}_{(2)}} g_{(2)}\\
&= Y^1 x^1{}_{(1)} aS(Y^2 x^1{}_{(2)})\alpha Y^3 x^2 h_{(1)} X^1 y^1{}_{(1)} bS(X^2 y^1{}_{(2)})\alpha X^3 y^2 g_{(1)}\\
&\quad \otimes x^3 h_{(2)} y^3 g_{(2)}\\
&= (q^1(x^1 \triangleright a)S(q^2)x^2 h_{(1)})(Q^1(y^1 \triangleright b)S(Q^2)y^2 g_{(1)}) \otimes (x^3 h_{(2)})(y^3 g_{(2)})\\
&= \chi(a\otimes h)\chi(b\otimes g)
\end{aligned}
$$

Next we show that χ is a coalgebra morphism.

$$
\begin{aligned}
&(\chi\otimes\chi)\Delta(\chi^{-1}(b\otimes 1))\\
&= q^1(a^1 w^1 Y^1 \triangleright t^1 T^1 x^1{}_{(1)} b_{(1)} X^1{}_{(1)} \beta_{(1)} S(x^2 X^2)_{(1)} g^1 S(t^2 R'^{(2)} h^3 T^3{}_{(2)}))S(q^2)\\
&\quad a^2 w^2{}_{(1)} W^1{}_{(1)} \underline{R^{(2)}{}_{(1)}} y^2{}_{(1)} Y^3{}_{(1)(1)} x^3{}_{(1)(1)} X^3{}_{(1)(1)}\\
&\quad \otimes a^3 w^2{}_{(2)} W^1{}_{(2)} \underline{R^{(2)}{}_{(2)}} y^2{}_{(2)} Y^3{}_{(1)(2)} x^3{}_{(1)(2)} X^3{}_{(1)(2)}\\
&\quad \otimes Q^1(d^1 w^3{}_{(1)} W^2 \underline{R^{(1)}} y^1 Y^2 t^3 R'^{(1)} \triangleright h^1 T^2 x^1{}_{(2)} b_{(2)} \beta_{(2)} S(x^2 X^2)_{(2)} g^2 S(h^2 T^3{}_{(1)}))S(Q^2)\\
&\quad d^2 w^3{}_{(2)(1)} W^3{}_{(1)} y^3{}_{(1)} Y^3{}_{(2)(1)} x^3{}_{(2)(1)} X^3{}_{(2)(1)}\\
&\quad \otimes d^3 w^3{}_{(2)(2)} W^3{}_{(2)} y^3{}_{(2)} Y^3{}_{(2)(2)} x^3{}_{(2)(2)} X^3{}_{(2)(2)}\\
&= q^1(a^1 w^1 Y^1 \triangleright t^1 T^1 x^1{}_{(1)} b_{(1)} X^1{}_{(1)} \delta^1 S(t^2 R'^{(2)} h^3 T^3{}_{(2)} x^2{}_{(2)} X^2{}_{(2)}))S(q^2)\\
&\quad a^2 w^2{}_{(1)} W^1{}_{(1)} u^1 H^1 R''^{(2)} \underline{v^2 y^2{}_{(1)}} Y^3{}_{(1)(1)} x^3{}_{(1)(1)} X^3{}_{(1)(1)}\\
&\quad \otimes a^3 w^2{}_{(2)} W^1{}_{(2)} u^2 R^{(2)} H^3 \underline{v^3 y^2{}_{(2)}} Y^3{}_{(1)(2)} x^3{}_{(1)(2)} X^3{}_{(1)(2)}\\
&\quad \otimes Q^1(d^1 w^3{}_{(1)} W^2 u^3 R^{(1)} H^2 R''^{(1)} \underline{v^1 y^1} Y^2 t^3 R'^{(1)} \triangleright h^1 T^2 x^1{}_{(2)} b_{(2)} X^1{}_{(2)} \delta^2\\
&\quad S(h^2 T^3{}_{(1)} x^2{}_{(1)} X^t{}_{(1)}))S(Q^2) d^2 w^3{}_{(2)(1)} W^3{}_{(1)} \underline{y^3{}_{(1)}} Y^3{}_{(2)(1)} x^3{}_{(2)(1)} X^3{}_{(2)(1)}\\
&\quad \otimes d^3 w^3{}_{(2)(2)} W^3{}_{(2)} \underline{y^3{}_{(2)}} Y^3{}_{(2)(2)} x^3{}_{(2)(2)} X^3{}_{(2)(2)}\\
&= q^1(a^1 w^1 Y^1 \triangleright t^1 T^1 x^1{}_{(1)} b_{(1)} X^1{}_{(1)} \delta^1 S(t^2 R'^{(2)} h^3 T^3{}_{(2)} x^2{}_{(2)} X^2{}_{(2)}))S(q^2)\\
&\quad a^2 w^2{}_{(1)} W^1{}_{(1)} u^1 \underline{H^1 R''^{(2)} y^1{}_{(2)}} v^2 \underline{D^1 Y^3{}_{(1)(1)} x^3{}_{(1)(1)} X^3{}_{(1)(1)}}\\
&\quad \otimes a^3 w^2{}_{(2)} W^1{}_{(2)} u^2 R^{(2)} \underline{H^3 y^2} v^3{}_{(1)} \underline{D^2 Y^3{}_{(1)(2)} x^3{}_{(1)(2)} X^3{}_{(1)(2)}}\\
&\quad \otimes Q^1(d^1 w^3{}_{(1)} W^2 u^3 R^{(1)} \underline{H^2 R''^{(1)} y^1{}_{(1)}} v^1 Y^2 Y^2 t^3 R'^{(1)} \triangleright h^1 T^2 x^1{}_{(2)} b_{(2)} X^1{}_{(2)} \delta^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}\underline{y^3{}_{(1)}}v^3{}_{(2)(1)}\underline{D^3{}_{(1)}Y^3{}_{(2)(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}\underline{y^3{}_{(2)}}v^3{}_{(2)(2)}\underline{D^3{}_{(2)}Y^3{}_{(2)(2)}x^3{}_{(2)(2)}X^3{}_{(2)(2)}}\\
&= q^1(a^1{}_{(1)}w^1{}_{(1)}\underline{Y^1{}_{(1)}}t^1T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(a^1{}_{(2)}w^1{}_{(2)}\underline{Y^1{}_{(2)}}t^2R'^{(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))\\
&\quad S(q^2)a^2w^2{}_{(1)}W^1{}_{(1)}u^1y^1H^1R''^{(2)}\underline{v^2Y^3{}_{(1)}}x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^3w^2{}_{(2)}W^1{}_{(2)}u^2\underline{R^{(2)}y^2{}_{(2)}}z^2H^3{}_{(1)}\underline{v^3{}_{(1)}Y^3{}_{(2)(1)}}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1w^3{}_{(1)}W^2u^3\underline{R^{(1)}y^2{}_{(1)}}z^1H^2R''^{(1)}\underline{v^1Y^2}t^3R'^{(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}\underline{v^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}}\\
&\quad x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}\underline{v^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= q^1(a^1{}_{(1)}w^1{}_{(1)}Y^1{}_{(1)}F^1{}_{(1)}\underline{v^1{}_{(1)(1)}t^1}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(a^1{}_{(2)}w^1{}_{(2)}Y^1{}_{(2)}F^1{}_{(2)}\underline{v^1{}_{(1)(2)}t^2R'^{(2)}}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))S(q^2)\\
&\quad a^2w^2{}_{(1)}W^1{}_{(1)}u^1y^1H^1\underline{R''^{(2)}Y^2{}_{(2)}}F^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^3w^2{}_{(2)}W^1{}_{(2)}u^2y^2{}_{(1)}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1w^3{}_{(1)}W^2u^3y^2{}_{(2)}R^{(1)}z^1H^2\underline{R''^{(1)}Y^2{}_{(1)}}F^2\underline{v^1{}_{(2)}t^3R'^{(1)}}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}\\
&\quad Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= q^1(a^1{}_{(1)}w^1{}_{(1)}Y^1{}_{(1)}\underline{F^1{}_{(1)}t^1}v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(a^1{}_{(2)}w^1{}_{(2)}Y^1{}_{(2)}\underline{F^1{}_{(2)}t^2}R'^{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))S(q^2)\\
&\quad a^2w^2{}_{(1)}W^1{}_{(1)}u^1y^1H^1Y^2{}_{(1)}R''^{(2)}\underline{F^3}v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^3w^2{}_{(2)}W^1{}_{(2)}u^2y^2{}_{(1)}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1w^3{}_{(1)}W^2u^3y^2{}_{(2)}R^{(1)}z^1H^2Y^2{}_{(2)}R''^{(1)}\underline{F^2t^3}R'^{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}\\
&\quad Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= q^1(a^1{}_{(1)}w^1{}_{(1)}Y^1{}_{(1)}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(a^1{}_{(2)}w^1{}_{(2)}Y^1{}_{(2)}t^2F^1\underline{U^2{}_{(1)}R'^{(2)}}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))S(q^2)\\
&\quad a^2w^2{}_{(1)}W^1{}_{(1)}u^1y^1H^1Y^2{}_{(1)}\underline{R''^{(2)}t^3{}_{(2)}}F^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^3w^2{}_{(2)}W^1{}_{(2)}u^2y^2{}_{(1)}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1w^3{}_{(1)}W^2u^3y^2{}_{(2)}R^{(1)}z^1H^2Y^2{}_{(2)}\underline{R''^{(1)}t^3{}_{(1)}}F^2\underline{U^2{}_{(2)}R'^{(1)}}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}\\
&\quad Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&= q^1(a^1{}_{(1)}w^1{}_{(1)}Y^1{}_{(1)}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(a^1{}_{(2)}w^1{}_{(2)}Y^1{}_{(2)}t^2\underline{F^1R'^{(2)}}U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))S(q^2)\\
&\quad a^2w^2{}_{(1)}W^1{}_{(1)}u^1y^1H^1Y^2{}_{(1)}t^3{}_{(1)}\underline{R''^{(2)}F^3}U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^3w^2{}_{(2)}W^1{}_{(2)}u^2y^2{}_{(1)}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1w^3{}_{(1)}W^2u^3y^2{}_{(2)}R^{(1)}z^1H^2Y^2{}_{(2)}t^3{}_{(2)}\underline{R''^{(1)}F^2R'^{(1)}}U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2w^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}\\
&\quad Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3w^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= q^1(\underline{a^1{}_{(1)}w^1{}_{(1)}}Y^1{}_{(1)}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(\underline{a^1{}_{(2)}w^1{}_{(2)}}Y^1{}_{(2)}t^2F^1R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)}))S(q^2)\\
&\quad \underline{a^2w^2{}_{(1)}}W^1{}_{(1)}u^1y^1H^1Y^2{}_{(1)}t^3{}_{(1)}F^2R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes\underline{a^3w^2{}_{(2)}}W^1{}_{(2)}u^2y^2{}_{(1)}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1\underline{w^3{}_{(1)}}W^2u^3y^2{}_{(2)}R^{(1)}z^1H^2Y^2{}_{(2)}t^3{}_{(2)}F^3R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)\\
&\quad d^2\underline{w^3{}_{(2)(1)}}W^3{}_{(1)}y^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3\underline{w^3{}_{(2)(2)}}W^3{}_{(2)}y^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= \underline{B^1a^1{}_{(1)(1)}}w^1{}_{(1)}Y^1{}_{(1)}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(\underline{B^2a^1{}_{(1)(2)}}w^1{}_{(2)}Y^1{}_{(2)}t^2F^1R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha\\
&\quad \underline{B^3a^1{}_{(2)}}w^2\underline{G^1W^1{}_{(1)}u^1y^1}H^1Y^2{}_{(1)}t^3{}_{(1)}F^2R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^2w^3{}_{(1)}\underline{G^2W^1{}_{(2)}u^2y^2{}_{(1)}}R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1a^3{}_{(1)}w^3{}_{(2)(1)}\underline{G^3{}_{(1)}W^2u^3y^2{}_{(2)}}R^{(1)}z^1H^2Y^2{}_{(2)}t^3{}_{(2)}F^3R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1\\
&\quad T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2a^3{}_{(2)(1)}w^3{}_{(2)(2)(1)}\\
&\quad \underline{G^3{}_{(2)(1)}W^3{}_{(1)}y^3{}_{(1)}}z^3{}_{(1)}H^3{}_{(2)(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3a^3{}_{(2)(2)}w^3{}_{(2)(2)(2)}\underline{G^3{}_{(2)(2)}W^3{}_{(2)}y^3{}_{(2)}}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1B^1\underline{w^1{}_{(1)}Y^1{}_{(1)}}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1\\
&\quad S(B^2\underline{w^1{}_{(2)}Y^1{}_{(2)}}t^2F^1R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha\\
&\quad B^3\underline{w^2H^1Y^2{}_{(1)}}t^3{}_{(1)}F^2R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1\\
&\quad \otimes a^2\underline{w^3{}_{(1)}G^1R^{(2)}z^2H^3{}_{(1)}Y^3{}_{(1)}}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2\\
&\quad \otimes Q^1(d^1a^3{}_{(1)}\underline{w^3{}_{(2)(1)}G^2R^{(1)}z^1H^2Y^2{}_{(2)}}t^3{}_{(2)}F^3R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2a^3{}_{(2)(1)}\\
&\quad \underline{w^3{}_{(2)(2)(1)}G^3{}_{(1)}z^3{}_{(1)}H^3{}_{(2)(1)}Y^3{}_{(2)(1)}}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes d^3a^3{}_{(2)(2)}\underline{w^3{}_{(2)(2)(2)}G^3{}_{(2)}z^3{}_{(2)}H^3{}_{(2)(2)}Y^3{}_{(2)(2)}}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1\underline{B^1Y^1{}_{(1)(1)}}w^1{}_{(1)}t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1
\end{aligned}
$$

$$\begin{aligned}
&\quad S(\underline{B^2Y^1{}_{(1)(2)}}w^1{}_{(2)}t^2F^1R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha \\
&\quad \underline{B^3Y^1{}_{(2)}}w^2t^3{}_{(1)}F^2R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1 \\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2 \\
&\otimes Q^1(d^1a^3{}_{(1)}G^2R^{(1)}z^1Y^2w^3t^3{}_{(2)}F^3R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2 \\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2a^3{}_{(2)(1)} \\
&\quad G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)} \\
&\otimes d^3a^3{}_{(2)(2)}G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)} \\
= {}& a^1Y^1\underline{B^1w^1{}_{(1)}t^1}U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1 \\
&\quad S(\underline{B^2w^1{}_{(2)}t^2F^1}R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha \\
&\quad \underline{B^3w^2t^3{}_{(1)}F^2}R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1 \\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2 \\
&\otimes Q^1(d^1a^3{}_{(1)}G^2R^{(1)}z^1Y^2\underline{w^3t^3{}_{(2)}F^3}R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2 \\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2a^3{}_{(2)(1)}G^3{}_{(1)}z^3{}_{(1)} \\
&\quad Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)} \\
&\otimes d^3a^3{}_{(2)(2)}G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)} \\
= {}& a^1Y^1t^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(t^2{}_{(1)}R'^{(2)}{}_{(1)}A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha \\
&\quad t^2{}_{(2)}R'^{(2)}{}_{(2)}A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1 \\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2 \\
&\otimes Q^1(d^1a^3{}_{(1)}G^2R^{(1)}z^1Y^2t^3R'^{(1)}A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2 \\
&\quad S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2)d^2a^3{}_{(2)(1)}G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)} \\
&\otimes d^3a^3{}_{(2)(2)}G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)} \\
= {}& a^1Y^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha \\
&\quad A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1 \\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2 \\
&\otimes Q^1(\underline{d^1a^3{}_{(1)}}G^2R^{(1)}z^1Y^2A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2S(h^2T^3{}_{(1)}x^2{}_{(1)}X^t{}_{(1)}))S(Q^2) \\
&\quad \underline{d^2a^3{}_{(2)(1)}}G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)} \\
&\otimes \underline{d^3a^3{}_{(2)(2)}}G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)} \\
= {}& a^1Y^1U^1v^1{}_{(1)}T^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(A^2U^2{}_{(2)}v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}X^2{}_{(2)})\alpha \\
&\quad A^3U^3v^2x^3{}_{(1)}X^3{}_{(1)}D^1 \\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}v^3{}_{(1)}x^3{}_{(2)(1)}X^3{}_{(2)(1)}D^2 \\
&\otimes a^3{}_{(1)}Q^1(d^1G^2R^{(1)}z^1Y^2A^1U^2{}_{(1)}v^1{}_{(2)(1)}\triangleright h^1T^2x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2S(h^2T^3{}_{(1)}x^2{}_{(1)}X^2{}_{(1)}))S(Q^2) \\
&\quad d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)} \\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)} \\
= {}& a^1Y^1U^1\underline{v^1{}_{(1)}T^1x^1{}_{(1)}}b_{(1)}X^1{}_{(1)}\delta^1S(A^2U^2{}_{(2)}\underline{v^1{}_{(2)(2)}h^3T^3{}_{(2)}x^2{}_{(2)}}X^2{}_{(2)})\alpha
\end{aligned}$$

$$
\begin{aligned}
&A^3U^3\underline{v^2x^3{}_{(1)}}X^3{}_{(1)}D^1\\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}\underline{v^3{}_{(1)}x^3{}_{(2)(1)}}X^3{}_{(2)(1)}D^2\\
&\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}Y^2{}_{(1)}A^1{}_{(1)}U^2{}_{(1)(1)}\underline{v^1{}_{(2)(1)(1)}h^1T^2x^1{}_{(2)}}b_{(2)}X^1{}_{(2)}\delta^2\\
&S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}Y^2{}_{(2)}A^1{}_{(2)}U^2{}_{(1)(2)}\underline{v^1{}_{(2)(1)(2)}h^2T^3{}_{(1)}x^2{}_{(1)}}X^2{}_{(1)})\alpha\\
&H^3d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}\underline{v^3{}_{(2)(1)}x^3{}_{(2)(2)(1)}}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}\underline{v^3{}_{(2)(2)}x^3{}_{(2)(2)(2)}}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
=\,&a^1Y^1U^1T^1v^1{}_{(1)}x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(A^2\underline{U^2{}_{(2)}h^3}T^3{}_{(2)}v^2{}_{(2)}x^2{}_{(1)(2)}w^1{}_{(2)}X^2{}_{(2)})\alpha\\
&A^3U^3v^3x^2{}_{(2)}w^2X^3{}_{(1)}D^1\\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}x^3{}_{(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}Y^2{}_{(1)}A^1{}_{(1)}\underline{U^2{}_{(1)(1)}h^1}T^2v^1{}_{(2)}x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}Y^2{}_{(2)}A^1{}_{(2)}\underline{U^2{}_{(1)(2)}h^2}T^3{}_{(1)}v^2{}_{(1)}x^2{}_{(1)(1)}w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&H^3d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}x^3{}_{(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}x^3{}_{(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
=\,&a^1Y^1U^1T^1v^1{}_{(1)}x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(\underline{A^2h^3}U^2{}_{(2)(2)}T^3{}_{(2)}v^2{}_{(2)}x^2{}_{(1)(2)}w^1{}_{(2)}X^2{}_{(2)})\alpha\\
&\underline{A^3}U^3v^3x^2{}_{(2)}w^2X^3{}_{(1)}D^1\\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}x^3{}_{(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}Y^2{}_{(1)}\underline{A^1{}_{(1)}h^1}U^2{}_{(1)}T^2v^1{}_{(2)}x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}Y^2{}_{(2)}\underline{A^1{}_{(2)}h^2}U^2{}_{(2)(1)}T^3{}_{(1)}v^2{}_{(1)}x^2{}_{(1)(1)}w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&H^3d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}x^3{}_{(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}x^3{}_{(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
=\,&a^1Y^1\underline{U^1T^1v^1{}_{(1)}}x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(B^2\underline{A^2{}_{(2)}U^2{}_{(2)(2)}T^3{}_{(2)}v^2{}_{(2)}}x^2{}_{(1)(2)}w^1{}_{(2)}X^2{}_{(2)})\alpha\\
&B^3\underline{A^3U^3v^3}x^2{}_{(2)}w^2X^3{}_{(1)}D^1\\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}x^3{}_{(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}Y^2{}_{(1)}\underline{A^1U^2{}_{(1)}T^2v^1{}_{(2)}}x^1{}_{(2)}b_{(2)}X^1{}_{(2)}\delta^2\\
&S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}Y^2{}_{(2)}B^1\underline{A^2{}_{(1)}U^2{}_{(2)(1)}T^3{}_{(1)}v^2{}_{(1)}}x^2{}_{(1)(1)}w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&H^3d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}x^3{}_{(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}x^3{}_{(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
=\,&a^1Y^1U^1x^1{}_{(1)}b_{(1)}X^1{}_{(1)}\delta^1S(\underline{B^2U^3{}_{(1)(2)}x^2{}_{(1)(2)}}w^1{}_{(2)}X^2{}_{(2)})\alpha\underline{B^3U^3{}_{(2)}x^2{}_{(2)}}w^2X^3{}_{(1)}D^1\\
&\otimes a^2G^1R^{(2)}z^2Y^3{}_{(1)}x^3{}_{(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}Y^2{}_{(1)}U^2x^1{}_{(2)}b_{(2)}\delta^2S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}Y^2{}_{(2)}\\
&\underline{B^1U^3{}_{(1)(1)}x^2{}_{(1)(1)}}w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&H^3d^2G^3{}_{(1)}z^3{}_{(1)}Y^3{}_{(2)(1)}x^3{}_{(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}Y^3{}_{(2)(2)}x^3{}_{(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&= a^1\underline{Y^1U^1x^1{}_{(1)}}b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2G^1R^{(2)}z^2\underline{Y^3{}_{(1)}x^3{}_{(1)}}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}H^1d^1{}_{(1)}G^2{}_{(1)}R^{(1)}{}_{(1)}z^1{}_{(1)}\underline{Y^2{}_{(1)}U^2x^1{}_{(2)}}b_{(2)}X^1{}_{(2)}\delta^2S(H^2d^1{}_{(2)}G^2{}_{(2)}R^{(1)}{}_{(2)}z^1{}_{(2)}\\
&\quad\quad\underline{Y^2{}_{(2)}U^3x^2}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad H^3d^2G^3{}_{(1)}z^3{}_{(1)}\underline{Y^3{}_{(2)(1)}x^3{}_{(2)(1)}}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}d^3G^3{}_{(2)}z^3{}_{(2)}\underline{Y^3{}_{(2)(2)}x^3{}_{(2)(2)}}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2\underline{G^1}R^{(2)}z^2x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}H^1\underline{d^1{}_{(1)}G^2{}_{(1)}}R^{(1)}{}_{(1)}z^1{}_{(1)}x^1Y^2b_{(2)}\delta^2S(H^2\underline{d^1{}_{(2)}G^2{}_{(2)}}R^{(1)}{}_{(2)}z^1{}_{(2)}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad H^3\underline{d^2G^3{}_{(1)}}z^3{}_{(1)}x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}\underline{d^3G^3{}_{(2)}}z^3{}_{(2)}x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2G^1A^1\underline{d^1{}_{(1)}R^{(2)}}z^2x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}H^1G^2{}_{(1)(1)}A^2{}_{(1)}\underline{d^1{}_{(2)(1)}R^{(1)}{}_{(1)}}z^1{}_{(1)}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2S(H^2G^2{}_{(1)(2)}A^2{}_{(2)}\\
&\quad\quad\underline{d^1{}_{(2)(2)}R^{(1)}{}_{(2)}}z^1{}_{(2)}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad H^3G^2{}_{(2)}A^3d^2z^3{}_{(1)}x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3d^3z^3{}_{(2)}x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2G^1A^1R^{(2)}d^1{}_{(2)}z^2x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}\underline{H^1G^2{}_{(1)(1)}}A^2{}_{(1)}R^{(1)}{}_{(1)}d^1{}_{(1)(1)}z^1{}_{(1)}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(\underline{H^2G^2{}_{(1)(2)}}A^2{}_{(2)}R^{(1)}{}_{(2)}d^1{}_{(1)(2)}z^1{}_{(2)}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad \underline{H^3G^2{}_{(2)}}A^3d^2z^3{}_{(1)}x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3d^3z^3{}_{(2)}x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2G^1\underline{A^1}R^{(2)}d^1{}_{(2)}z^2x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}G^2\underline{H^1A^2{}_{(1)}}R^{(1)}{}_{(1)}d^1{}_{(1)(1)}z^1{}_{(1)}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(\underline{H^2A^2{}_{(2)}}R^{(1)}{}_{(2)}d^1{}_{(1)(2)}z^1{}_{(2)}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad \underline{H^3A^3}d^2z^3{}_{(1)}x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3d^3z^3{}_{(2)}x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad\otimes a^2G^1A^1{}_{(1)}y^1R^{(2)}\underline{d^1{}_{(2)}z^2}x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}D^2\\
&\quad\otimes a^3{}_{(1)}G^2A^1{}_{(2)}y^2R^{(1)}{}_{(1)}\underline{d^1{}_{(1)(1)}z^1{}_{(1)}}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2y^3R^{(1)}{}_{(2)}\underline{d^1{}_{(1)(2)}z^1{}_{(2)}}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\quad A^3\underline{d^2z^3{}_{(1)}}x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3\underline{d^3z^3{}_{(2)}}x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad \otimes a^2G^1A^1{}_{(1)}y^1R^{(2)}d^2z^2{}_{(1)}\underline{h^1x^3{}_{(1)}Y^3{}_{(2)(1)}w^3{}_{(1)}X^3{}_{(2)(1)}}D^2\\
&\quad \otimes a^3{}_{(1)}G^2A^1{}_{(2)}y^2R^{(1)}{}_{(1)}d^1{}_{(1)}z^1{}_{(1)}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2y^3R^{(1)}{}_{(2)}d^1{}_{(2)}z^1{}_{(2)}x^2Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad A^3d^3z^2{}_{(2)}\underline{h^2x^3{}_{(2)(1)}Y^3{}_{(2)(2)(1)}w^3{}_{(2)(1)}X^3{}_{(2)(2)(1)}}D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3z^3\underline{h^3x^3{}_{(2)(2)}Y^3{}_{(2)(2)(2)}w^3{}_{(2)(2)}X^3{}_{(2)(2)(2)}}D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad \otimes a^2G^1A^1{}_{(1)}y^1R^{(2)}d^2\underline{z^2{}_{(1)}x^3{}_{(1)(1)}}Y^3{}_{(2)(1)(1)}w^3{}_{(1)(1)}X^3{}_{(2)(1)(1)}h^1D^2\\
&\quad \otimes a^3{}_{(1)}G^2A^1{}_{(2)}y^2R^{(1)}{}_{(1)}d^1{}_{(1)}\underline{z^1{}_{(1)}x^1}Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2y^3R^{(1)}{}_{(2)}d^1{}_{(2)}\underline{z^1{}_{(2)}x^2}Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad A^3d^3\underline{z^2{}_{(2)}x^3{}_{(1)(2)}}Y^3{}_{(2)(1)(2)}w^3{}_{(1)(2)}X^3{}_{(2)(1)(2)}h^2D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3\underline{z^3x^3{}_{(2)}}Y^3{}_{(2)(2)}w^3{}_{(2)}X^3{}_{(2)(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad \otimes a^2G^1A^1{}_{(1)}y^1R^{(2)}\underline{d^2z^3{}_{(1)}}x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}w^3{}_{(1)(1)}X^3{}_{(2)(1)(1)}h^1D^2\\
&\quad \otimes a^3{}_{(1)}G^2A^1{}_{(2)}y^2R^{(1)}{}_{(1)}\underline{d^1{}_{(1)}z^1}x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2y^3R^{(1)}{}_{(2)}\underline{d^1{}_{(2)}z^2}x^2{}_{(1)}k^1Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad A^3\underline{d^3z^3{}_{(2)}}x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}w^3{}_{(1)(2)}X^3{}_{(2)(1)(2)}h^2D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}w^3{}_{(2)}X^3{}_{(2)(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad \otimes a^2G^1A^1{}_{(1)}\underline{y^1R^{(2)}d^3}z^2{}_{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}w^3{}_{(1)(1)}X^3{}_{(2)(1)(1)}h^1D^2\\
&\quad \otimes a^3{}_{(1)}G^2A^1{}_{(2)}\underline{y^2R^{(1)}{}_{(1)}d^1}z^1x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2\underline{y^3R^{(1)}{}_{(2)}d^2}z^2{}_{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad A^3z^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}w^3{}_{(1)(2)}X^3{}_{(2)(1)(2)}h^2D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}w^3{}_{(2)}X^3{}_{(2)(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1\\
&\quad \otimes a^2G^1\underline{A^1{}_{(1)}R^{(2)}}y^2\underline{R'^{(2)}z^2{}_{(2)}}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}w^3{}_{(1)(1)}X^3{}_{(2)(1)(1)}h^1D^2\\
&\quad \otimes a^3{}_{(1)}G^2\underline{A^1{}_{(2)}R^{(1)}}y^1z^1x^1Y^2b_{(2)}X^1{}_{(2)}\delta^2\\
&\quad\quad S(A^2y^3\underline{R'^{(1)}z^2{}_{(1)}}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\\
&\quad\quad A^3z^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}w^3{}_{(1)(2)}X^3{}_{(2)(1)(2)}h^2D^3{}_{(1)}\\
&\quad \otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}w^3{}_{(2)}X^3{}_{(2)(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}D^1
\end{aligned}
$$

$$
\begin{aligned}
&\otimes a^2 G^1 R^{(2)} \underline{A^1_{(2)} y^2 z^2_{(1)}} R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} w^3_{(1)(1)} X^3_{(2)(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} \underline{A^1_{(1)} y^1 z^1} x^1 Y^2 b_{(2)} X^1_{(2)} \delta^2 S(\underline{A^2 y^3 z^2_{(2)}} R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} B^1 w^1_{(1)} X^2_{(1)})\alpha \\
&\quad \underline{A^3 z^3} u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} w^3_{(1)(2)} X^3_{(2)(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} w^3_{(2)} X^3_{(2)(2)} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} X^1_{(1)} \delta^1 S(B^2 w^1_{(2)} X^2_{(2)}) \alpha B^3 w^2 X^3_{(1)} D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} w^3_{(1)(1)} X^3_{(2)(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} X^1_{(2)} \delta^2 S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} B^1 w^1_{(1)} X^2_{(1)})\alpha \\
&\quad A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} w^3_{(1)(2)} X^3_{(2)(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} w^3_{(2)} X^3_{(2)(2)} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} \underline{X^1_{(1)}} \delta^1 S(B^2 \underline{w^1_{(2)} X^2_{(2)}}) \alpha B^3 \underline{w^2 X^3_{(1)}} D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} \underline{w^3_{(1)(1)} X^3_{(2)(1)(1)}} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} \underline{X^1_{(2)}} \delta^2 S(t^1 B^1 \underline{w^1_{(1)} X^2_{(1)}})\alpha t^2 \beta S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} t^3)\alpha \\
&\quad A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} \underline{w^3_{(1)(2)} X^3_{(2)(1)(2)}} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} \underline{w^3_{(2)} X^3_{(2)(2)}} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} W^1_{(1)} X^1_{(1)} \delta^1 S(\underline{B^2 W^2_{(1)(2)}} X^2_{(2)}) \alpha \underline{B^3 W^2_{(2)}} X^3 D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} W^3_{(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} W^1_{(2)} X^1_{(2)} \delta^2 S(t^1 \underline{B^1 W^2_{(1)(1)}} X^2_{(1)})\alpha t^2 \beta S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} t^3)\alpha \\
&\quad A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} W^3_{(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} W^3_{(2)} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} W^1_{(1)} X^1_{(1)} \delta^1 S(B^2 X^2_{(2)}) \alpha B^3 X^3 D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} W^3_{(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} W^1_{(2)} X^1_{(2)} \delta^2 S(\underline{t^1 W^2} B^1 X^2_{(1)})\alpha \underline{t^2} \beta S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} \underline{t^3})\alpha \\
&\quad A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} W^3_{(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} W^3_{(2)} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} W^1_{(1)} V^1_{(1)} X^1_{(1)} \delta^1 S(B^2 X^2_{(2)}) \alpha B^3 X^3 \underline{T^1} d^1 D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} W^3_{(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} W^1_{(2)} V^1_{(2)} X^1_{(2)} \delta^2 S(t^1 V^2 B^1 X^2_{(1)})\alpha t^2 V^3_{(1)} T^2 d^2 \beta \\
&\quad S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} W^2 t^3 V^3_{(2)} T^3 d^3)\alpha A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} W^3_{(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} W^3_{(2)} h^3 D^3_{(2)} \\
= {} & a^1 Y^1 b_{(1)} W^1_{(1)} V^1_{(1)} \underline{X^1_{(1)} T^1_{(1)(1)(1)}} \delta^1 S(B^2 \underline{X^2_{(2)} T^1_{(1)(2)(2)}}) \alpha B^3 \underline{X^3 T^1_{(2)}} d^1 D^1 \\
&\otimes a^2 G^1 R^{(2)} A^1 R'^{(2)} u^2 x^2_{(2)(1)} k^2_{(1)} Y^3_{(2)(1)(1)} W^3_{(1)(1)} h^1 D^2 \\
&\otimes a^3_{(1)} G^2 R^{(1)} x^1 Y^2 b_{(2)} W^1_{(2)} V^1_{(2)} \underline{X^1_{(2)} T^1_{(1)(1)(2)}} \delta^2 S(t^1 V^2 B^1 \underline{X^2_{(1)} T^1_{(1)(2)(1)}})\alpha t^2 V^3_{(1)} T^2 d^2 \beta \\
&\quad S(A^2 R'^{(1)} u^1 x^2_{(1)} k^1 Y^3_{(1)} W^2 t^3 V^3_{(2)} T^3 d^3)\alpha A^3 u^3 x^2_{(2)(2)} k^2_{(2)} Y^3_{(2)(1)(2)} W^3_{(1)(2)} h^2 D^3_{(1)} \\
&\otimes a^3_{(2)} G^3 x^3 k^3 Y^3_{(2)(2)} W^3_{(2)} h^3 D^3_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&= a^1Y^1b_{(1)}W^1{}_{(1)}V^1{}_{(1)}T^1{}_{(1)(1)}X^1{}_{(1)}\delta^1S(\underline{B^2T^1{}_{(2)(1)(2)}}X^2{}_{(2)})\alpha\underline{B^3T^1{}_{(2)(2)}}X^3d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}V^1{}_{(2)}T^1{}_{(1)(2)}X^1{}_{(2)}\delta^2S(t^1V^2\underline{B^1T^1{}_{(2)(1)(1)}}X^2{}_{(1)})\alpha t^2V^3{}_{(1)}T^2d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2t^3V^3{}_{(2)}T^3d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}W^1{}_{(1)}V^1{}_{(1)}T^1{}_{(1)(1)}\underline{X^1{}_{(1)}}\delta^1S(B^2\underline{X^2{}_{(2)}})\alpha B^3\underline{X^3}d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}V^1{}_{(2)}T^1{}_{(1)(2)}\underline{X^1{}_{(2)}}\delta^2S(t^1V^2T^1{}_{(2)}B^1\underline{X^2{}_{(1)}})\alpha t^2V^3{}_{(1)}T^2d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2t^3V^3{}_{(2)}T^3d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}W^1{}_{(1)}V^1{}_{(1)}T^1{}_{(1)(1)}z^1{}_{(1)}X^1{}_{(1)}H^1{}_{(1)(1)}\delta^1\\
&\qquad S(\underline{B^2z^2{}_{(1)(2)}}w^1{}_{(2)}X^2{}_{(2)}H^1{}_{(2)(2)})\alpha\underline{B^3z^2{}_{(2)}}w^2X^3{}_{(1)}H^2d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}V^1{}_{(2)}T^1{}_{(1)(2)}z^1{}_{(2)}X^1{}_{(2)}H^1{}_{(1)(2)}\delta^2\\
&\qquad S(t^1V^2T^1{}_{(2)}\underline{B^1z^2{}_{(1)(1)}}w^1{}_{(1)}X^2{}_{(1)}H^1{}_{(2)(1)})\alpha t^2V^3{}_{(1)}T^2z^3w^3X^3{}_{(2)}H^3d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2t^3V^3{}_{(2)}T^3d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}W^1{}_{(1)}\underline{V^1{}_{(1)}T^1{}_{(1)(1)}z^1{}_{(1)}}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}\underline{V^1{}_{(2)}T^1{}_{(1)(2)}z^1{}_{(2)}}X^1{}_{(2)}\delta^2\\
&\qquad S(\underline{t^1V^2T^1{}_{(2)}z^2}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha\underline{t^2V^3{}_{(1)}T^2z^3}w^3X^3{}_{(2)}d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2\underline{t^3V^3{}_{(2)}T^3}d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}W^1{}_{(1)}T^1{}_{(1)}X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}T^1{}_{(2)}X^1{}_{(2)}\delta^2S(T^2{}_{(1)}B^1w^1{}_{(1)}X^2{}_{(1)})\alpha T^2{}_{(2)}w^3X^3{}_{(2)}d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2T^3d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}W^1{}_{(1)}\underline{X^1{}_{(1)}\delta^1S(B^2w^1{}_{(2)}X^2{}_{(2)})\alpha B^3w^2X^3{}_{(1)}}d^1D^1\\
&\quad\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2x^2{}_{(2)(1)}k^2{}_{(1)}Y^3{}_{(2)(1)(1)}W^3{}_{(1)(1)}h^1D^2\\
&\quad\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}W^1{}_{(2)}\underline{X^1{}_{(2)}\delta^2S(B^1w^1{}_{(1)}X^2{}_{(1)})\alpha w^3X^3{}_{(2)}}d^2\beta\\
&\qquad S(A^2R'^{(1)}u^1x^2{}_{(1)}k^1Y^3{}_{(1)}W^2d^3)\alpha A^3u^3x^2{}_{(2)(2)}k^2{}_{(2)}Y^3{}_{(2)(1)(2)}W^3{}_{(1)(2)}h^2D^3{}_{(1)}\\
&\quad\otimes a^3{}_{(2)}G^3x^3k^3Y^3{}_{(2)(2)}W^3{}_{(2)}h^3D^3{}_{(2)}\\
&= a^1Y^1b_{(1)}\underline{W^1{}_{(1)}}d^1D^1
\end{aligned}
$$

$$
\begin{aligned}
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}{k^2}_{(1)}{Y^3}_{(2)(1)(1)}\underline{{W^3}_{(1)(1)}}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}\underline{{W^1}_{(2)}}d^2\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}k^1{Y^3}_{(1)}\underline{W^2}d^3)\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}{k^2}_{(2)}{Y^3}_{(2)(1)(2)}\underline{{W^3}_{(1)(2)}}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3k^3{Y^3}_{(2)(2)}\underline{{W^3}_{(2)}}h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}{X^1}_{(1)}{T^1}_{(1)}\underline{{t^1}_{(1)(1)}d^1}D^1\\
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}{k^2}_{(1)}{Y^3}_{(2)(1)(1)}{W^2}_{(1)}{X^2}_{(2)(1)}{T^3}_{(1)}{t^2}_{(1)}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}{X^1}_{(2)}{T^1}_{(2)}\underline{{t^1}_{(1)(2)}d^2}\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}k^1{Y^3}_{(1)}W^1{X^2}_{(1)}T^2\underline{{t^1}_{(2)}d^3})\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}{k^2}_{(2)}{Y^3}_{(2)(1)(2)}{W^2}_{(2)}{X^2}_{(2)(2)}{T^3}_{(2)}{t^2}_{(2)}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3k^3{Y^3}_{(2)(2)}W^3X^3t^3h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}{X^1}_{(1)}\underline{{T^1}_{(1)}d^1}t^1D^1\\
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}{k^2}_{(1)}{Y^3}_{(2)(1)(1)}{W^2}_{(1)}{X^2}_{(2)(1)}\underline{{T^3}_{(1)}}{t^2}_{(1)}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}{X^1}_{(2)}\underline{{T^1}_{(2)}d^2}\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}k^1{Y^3}_{(1)}W^1{X^2}_{(1)}\underline{T^2d^3})\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}{k^2}_{(2)}{Y^3}_{(2)(1)(2)}{W^2}_{(2)}{X^2}_{(2)(2)}\underline{{T^3}_{(2)}}{t^2}_{(2)}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3k^3{Y^3}_{(2)(2)}W^3X^3t^3h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}{X^1}_{(1)}d^1t^1D^1\\
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}\underline{{k^2}_{(1)}{Y^3}_{(2)(1)(1)}{W^2}_{(1)}}{X^2}_{(2)(1)}{d^3}_{(2)(1)}{T^3}_{(1)}{t^2}_{(1)}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}{X^1}_{(2)}d^2T^1\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}\underline{k^1{Y^3}_{(1)}W^1}{X^2}_{(1)}{d^3}_{(1)}T^2)\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}\underline{{k^2}_{(2)}{Y^3}_{(2)(1)(2)}{W^2}_{(2)}}{X^2}_{(2)(2)}{d^3}_{(2)(2)}{T^3}_{(2)}{t^2}_{(2)}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3\underline{k^3{Y^3}_{(2)(2)}W^3}X^3t^3h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}\underline{{X^1}_{(1)}d^1}t^1D^1\\
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}{Y^3}_{(1)(2)(1)}\underline{{X^2}_{(2)(1)}{d^3}_{(2)(1)}}{T^3}_{(1)}{t^2}_{(1)}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}\underline{{X^1}_{(2)}d^2}T^1\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}{Y^3}_{(1)(1)}\underline{{X^2}_{(1)}{d^3}_{(1)}}T^2)\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}{Y^3}_{(1)(2)(2)}\underline{{X^2}_{(2)(2)}{d^3}_{(2)(2)}}{T^3}_{(2)}{t^2}_{(2)}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3{Y^3}_{(2)}\underline{X^3}t^3h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}d^1\underline{H^1t^1}D^1\\
&\otimes a^2G^1R^{(2)}A^1R'^{(2)}u^2{x^2}_{(2)(1)}{Y^3}_{(1)(2)(1)}{d^3}_{(1)(2)(1)}{X^2}_{(2)(1)}\underline{{H^2}_{(2)(2)(1)}{T^3}_{(1)}{t^2}_{(1)}}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}d^2X^1\underline{{H^2}_{(1)}T^1}\beta S(A^2R'^{(1)}u^1{x^2}_{(1)}{Y^3}_{(1)(1)}{d^3}_{(1)(1)}{X^2}_{(1)}\underline{{H^2}_{(2)(1)}T^2})\alpha\\
&\quad A^3u^3{x^2}_{(2)(2)}{Y^3}_{(1)(2)(2)}{d^3}_{(1)(2)(2)}{X^2}_{(2)(2)}\underline{{H^2}_{(2)(2)(2)}{T^3}_{(2)}{t^2}_{(2)}}h^2{D^3}_{(1)}\\
&\otimes {a^3}_{(2)}G^3x^3{Y^3}_{(2)}{d^3}_{(2)}X^3\underline{H^3t^3}h^3{D^3}_{(2)}\\
=\,& a^1Y^1b_{(1)}d^1D^1\\
&\otimes a^2G^1R^{(2)}\underline{A^1R'^{(2)}u^2{x^2}_{(2)(1)}{Y^3}_{(1)(2)(1)}{d^3}_{(1)(2)(1)}{X^2}_{(2)(1)}}{T^3}_{(1)}h^1D^2\\
&\otimes {a^3}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}d^2X^1T^1\beta S(\underline{A^2R'^{(1)}u^1{x^2}_{(1)}{Y^3}_{(1)(1)}{d^3}_{(1)(1)}{X^2}_{(1)}}T^2)\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\underline{A^3u^3x^2{}_{(2)(2)}Y^3{}_{(1)(2)(2)}d^3{}_{(1)(2)(2)}X^2{}_{(2)(2)}}T^3{}_{(2)}h^2D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}G^3x^3Y^3{}_{(2)}d^3{}_{(2)}X^3h^3D^3{}_{(2)}\\
=&\, a^1Y^1b_{(1)}d^1D^1\\
&\otimes a^2G^1R^{(2)}x^2Y^3{}_{(1)}d^3{}_{(1)}X^2\underline{A^1R'^{(2)}u^2T^3{}_{(1)}}h^1D^2\\
&\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}d^2X^1\underline{T^1\beta S(A^2R'^{(1)}u^1T^2)\alpha A^3u^3T^3{}_{(2)}}h^2D^3{}_{(1)}\\
&\otimes a^3{}_{(2)}G^3x^3Y^3{}_{(2)}d^3{}_{(2)}X^3h^3D^3{}_{(2)}\\
=&\, a^1Y^1b_{(1)}d^1D^1\otimes a^2G^1R^{(2)}x^2Y^3{}_{(1)}d^3{}_{(1)}X^2R^{-(2)}h^1D^2\\
&\otimes a^3{}_{(1)}G^2R^{(1)}x^1Y^2b_{(2)}d^2X^1R^{-(1)}h^2D^3{}_{(1)}\otimes a^3{}_{(2)}G^3x^3Y^3{}_{(2)}d^3{}_{(2)}X^3h^3D^3{}_{(2)}\\
=&\, x^1Y^1b_{(1)}y^1X^1\otimes x^2T^1R^{(2)}w^2Y^3{}_{(1)}y^3{}_{(1)}W^2R^{-(2)}t^1X^2\\
&\otimes x^3{}_{(1)}T^2R^{(1)}w^1Y^2b_{(2)}y^2W^1R^{-(1)}t^2X^3{}_{(1)}\otimes x^3{}_{(2)}T^3w^3Y^3{}_{(2)}y^3{}_{(2)}W^3t^3X^3{}_{(2)}\\
=&\,\Delta(b\otimes 1)
\end{aligned}
$$

□

Example 5.2. Recall the structure of $\underline{D^\phi(G)}$ from section 3. We can now compute the structure of $\underline{D^\phi(G)}\rtimes D^\phi(G)$.

$$
\begin{aligned}
&((g\otimes\delta_s)\otimes(g'\otimes\delta_{s'}))((h\otimes\delta_t)\otimes(h'\otimes\delta_{t'}))=\\
&(g g' h g'^{-1}\otimes\delta_{gg'tg'^{-1}g^{-1}})\otimes(g'h'\otimes\delta_{g't'g'^{-1}})\,\delta_{s,gg'tg'^{-1}g^{-1}}\delta_{s',g'th^{-1}t^{-1}ht'g'^{-1}}\\
&\theta_{g't'g'^{-1}}(g',h')\theta_{gg'tg'^{-1}g^{-1}}(g,g'hg'^{-1})\gamma_{g'}(g'th^{-1}t^{-1}hg'^{-1},g't'g'^{-1})\\
&\gamma_{g'}(g'tg'^{-1},g'h^{-1}t^{-1}hg'^{-1})\theta^{-1}_{g'h^{-1}t^{-1}hg'^{-1}}(g',g'^{-1})\\
&\gamma^{-1}_{g'}(g'h^{-1}t^{-1}hg'^{-1},g'h^{-1}thg'^{-1})\theta_{g'tg'^{-1}}(g',h)\theta_{g'tg'^{-1}}(g'h,g'^{-1})\\
&\phi(gg'tg'^{-1}g^{-1},g't^{-1}g'^{-1},g'h^{-1}t^{-1}hg'^{-1})\phi^{-1}(g't^{-1}g'^{-1},g'tg'^{-1},g'h^{-1}t^{-1}hg'^{-1})\\
&\phi^{-1}(gg'tg'^{-1}g^{-1}g't^{-1}g'^{-1},g'th^{-1}t^{-1}hg'^{-1},g't'g'^{-1})
\end{aligned}
$$

$$\eta(1)=\sum_{s,t\in G}(e\otimes\delta_s)\otimes(e\otimes\delta_t)\,\phi(s^{-1},s,s^{-1})$$

$$
\begin{aligned}
&\Delta((g\otimes\delta_s)\otimes(g'\otimes\delta_{s'}))=\\
&\sum_{jk=s}\sum_{ab=t}(kgk^{-1}\otimes\delta_j)\otimes(kg^{-1}k^{-1}gg'\otimes\delta_{kg^{-1}k^{-1}gag^{-1}kgk^{-1}})\otimes(g\otimes\delta_k)\otimes(g'\otimes\delta_b)\\
&\gamma_{g'}(a,b)\gamma_g(j,k)\theta_j^{-1}(kgk^{-1},kg^{-1}k^{-1}g)\theta_{kg^{-1}k^{-1}gag^{-1}kgk^{-1}}(kg^{-1}k^{-1}g,g')\\
&\phi(s,g^{-1}s^{-1}g,g^{-1}sg)\phi^{-1}(j,kg^{-1}k^{-1}j^{-1}kgk^{-1},kg^{-1}k^{-1}jkgk^{-1})\phi^{-1}(k,g^{-1}k^{-1}g,g^{-1}kg)\\
&\phi^{-1}(jkg^{-1}k^{-1}j^{-1}kgk^{-1},kg^{-1}k^{-1}g,g^{-1}jkg)\phi^{-1}(kg^{-1}k^{-1}jkgk^{-1},kg^{-1}k^{-1}g,g^{-1}kg)\\
&\phi(kg^{-1}k^{-1}g,g^{-1}jg,g^{-1}kg)\phi(jkg^{-1}k^{-1}j^{-1}kgk^{-1},kg^{-1}k^{-1}jkgk^{-1},k)\\
&\phi(jkg^{-1}k^{-1}j^{-1}kgk^{-1},kg^{-1}k^{-1}g,ab)\phi(kg^{-1}k^{-1}gag^{-1}kgk^{-1},kg^{-1}k^{-1}g,b)\\
&\phi^{-1}(kg^{-1}k^{-1}g,a,b)\phi^{-1}(jkg^{-1}k^{-1}j^{-1}kgk^{-1},kg^{-1}k^{-1}gag^{-1}kgk^{-1},kg^{-1}k^{-1}gb)
\end{aligned}
$$

$$\varepsilon((g \otimes \delta_s) \otimes (g' \otimes \delta_{s'})) = \delta_{s,e}\delta_{s',e}$$

$$\begin{aligned}
&S((g \otimes \delta_s) \otimes (g' \otimes \delta_{s'})) = \\
&(g'^{-1}g^{-1}sg^{-1}s^{-1}gg' \otimes \delta_{g'^{-1}g^{-1}s^{-1}gg'}) \otimes (g'^{-1}g^{-1}sgs^{-1} \otimes \delta_{g'^{-1}g^{-1}sgs^{-1}s'^{-1}g'}) \\
&\theta^{-1}_{s^{-1}}(g, g^{-1})\gamma^{-1}_g(s, s^{-1})\theta^{-1}_{sg^{-1}s^{-1}gs'^{-1}g^{-1}sgs^{-1}}(sg^{-1}s^{-1}gg', g'^{-1}g^{-1}sgs^{-1}) \\
&\gamma^{-1}_{sg^{-1}s^{-1}gg'}(sg^{-1}s^{-1}gs'g^{-1}sgs^{-1}, sg^{-1}s^{-1}gs'^{-1}g^{-1}sgs^{-1})\theta_{sg^{-1}s^{-1}gs^{-1}}(sg^{-1}s^{-1}g, g^{-1}) \\
&\theta_{sg^{-1}s^{-1}gs'g^{-1}sgs^{-1}}(sg^{-1}s^{-1}g, g')\gamma_{g'^{-1}g^{-1}sgs^{-1}}(g'^{-1}sg^{-1}s^{-1}gg', g'^{-1}g^{-1}sgs^{-1}s'^{-1}g') \\
&\gamma_{g'^{-1}g^{-1}sgs^{-1}}(g'^{-1}g^{-1}s^{-1}gg', g'^{-1}g^{-1}sgsg^{-1}s^{-1}gg') \\
&\theta^{-1}_{g'^{-1}g^{-1}sgs^{-1}g^{-1}s^{-1}gg'}(g'^{-1}g^{-1}sgs^{-1}, sg^{-1}s^{-1}gg') \\
&\gamma^{-1}_{g'^{-1}g^{-1}sgs^{-1}}(g'^{-1}g^{-1}sgsg^{-1}s^{-1}gg', g'^{-1}g^{-1}sgs^{-1}g^{-1}s^{-1}gg') \\
&\theta_{g'^{-1}g^{-1}s^{-1}gg'}(g'^{-1}g^{-1}sgs^{-1}, sg^{-1}s^{-1})\phi(sg^{-1}s^{-1}gs'g^{-1}sgs^{-1}, sg^{-1}s^{-1}g, s'^{-1}g^{-1}sgs^{-1}) \\
&\phi(sg^{-1}s^{-1}gs^{-1}, sg^{-1}s^{-1}g, g^{-1}sg)\phi(s, g^{-1}s^{-1}g, g^{-1}sg)\phi^{-1}(sg^{-1}s^{-1}g, g^{-1}s^{-1}g, g^{-1}sg)
\end{aligned}$$

$$\alpha = \sum_{s,t \in G} (e \otimes \delta_s) \otimes (e \otimes \delta_t)\phi(s^{-1}, s, s^{-1})$$

$$\beta = \sum_{s,t \in G} (e \otimes \delta_s) \otimes (e \otimes \delta_t)\phi(s^{-1}, s, s^{-1})\phi(t^{-1}, t, t^{-1})$$

$$\begin{aligned}
\phi = \sum_{g,h,k \in G}\sum_{u,v,w \in G} &((e \otimes \delta_g) \otimes (e \otimes \delta_u)) \otimes ((e \otimes \delta_h) \otimes (e \otimes \delta_v)) \otimes ((e \otimes \delta_k) \otimes (e \otimes \delta_w)) \\
&\phi(u, v, w)\phi(g^{-1}, g, g^{-1})\phi(h^{-1}, h, h^{-1})\phi(k^{-1}, k, k^{-1})
\end{aligned}$$

The isomorphism for this example is

$$\begin{aligned}
\chi((g \otimes \delta_s) \otimes (h \otimes \delta_t)) = &(gh \otimes \delta_s) \otimes (h \otimes \delta_{g^{-1}s^{-1}gt})\theta_s(g, h)\gamma_h(g^{-1}sg, g^{-1}s^{-1}gt) \\
&\phi(s, g^{-1}s^{-1}g, g^{-1}sg)\phi^{-1}(sg^{-1}s^{-1}g, g^{-1}sg, g^{-1}s^{-1}gt)
\end{aligned}$$

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